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Several Derivative Formulas of Two Exponential Functions and Real Power of Hyperbolic Secant Function with a Generalization of a Formula for Specific Partial Bell Polynomials

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Abstract:

In the paper, by virtue of some identities for the partial Bell polynomials and with the aid of the Faá di Bruno formula, the author presents several derivative formulas of two exponential functions and the real power of the hyperbolic secant function, and generalizes a formula for specific partial Bell polynomials.

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1. INTRODUCTION

For $j\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$ and $z\in\mathbb{C}$, the falling factorial $\langle z\rangle_j$ and the rising factorial $(z)_j$ are respectively defined [1] by

$$\langle z\rangle_j=\prod_{\ell=0}^{j-1}(z-\ell)=\begin{cases} z(z-1)\cdots(z-j+1), & j\in\mathbb{N};\\ 1, & j=0\end{cases}$$

and $(z)_j=\prod_{\ell=0}^{j-1}(z+\ell)=\begin{cases} z(z+1)\cdots(z+j-1), & j\in\mathbb{N};\\ 1, & j=0.\end{cases}$

The rising factorial $(z)_j$ is also known as the Pochhammer symbol in literature.

The partial Bell polynomials $B_{j,k}$, also known as the Bell polynomials of the second kind, in combinatorics can be defined [2, p. 134, Theorem A] by

$$B_{j,k}(t_1,t_2,\dots,t_{j-k+1})=\sum_{\substack{1\le i\le j-k+1,\\ \ell_i\in\{0\}\cup\mathbb{N},\\ \sum_{i=1}^{j-k+1}i\ell_i=j,\\ \sum_{i=1}^{j-k+1}\ell_i=k}}\frac{j!}{\prod_{i=1}^{j-k+1}\ell_i!}\prod_{i=1}^{j-k+1}\left(\frac{t_i}{i!}\right)^{\ell_i}$$

for $j\ge k\in\mathbb{N}_0$. In [3], Hoffman's and Genčev's combinatorial identities were reformulated and generalized in terms of the partial Bell polynomials $B_{j,k}$. The identities and formulas

$$B_{j,k}(abt_1,ab^2t_2,\dots,ab^{j-k+1}t_{j-k+1})=a^kb^jB_{j,k}(t_1,t_2,\dots,t_{j-k+1}), \quad (1)$$

$$B_{j,k}(1!,2!,\dots,(j-k+1)!)=\binom{j-1}{k-1}\frac{j!}{k!}, \quad (2)$$

$$B_{j,k}(t_1,t_2,\dots,t_{j-k+1})=\sum_{\ell=0}^{k}\binom{j}{\ell}t_1^{\ell}B_{j-\ell,k-\ell}(0,t_2,t_3,\dots,t_{j-k+1}), \quad (3)$$

$$B_{j,k}\left(\frac{t_2}{2},\frac{t_3}{3},\dots,\frac{t_{j-k+2}}{j-k+2}\right)=\frac{j!}{(j+k)!}B_{j+k,k}(0,t_2,t_3,\dots,t_{j+1}), \quad (4)$$

$$B_{j,k}(0,0,\dots,0,t_\ell,0,\dots,0)=0, \quad except \quad B_{\ell k,k}=\frac{(\ell k)!}{k!(\ell!)^k}t_\ell^k, \quad (5)$$

and

$$B_{j,k}(t_1,t_2,\dots,t_{j-k+1})=\frac{j!}{(j-k)!}\sum_{\ell=0}^{k}\frac{t_1^{\ell}}{\ell!}B_{j-k,k-\ell}\left(\frac{t_2}{2},\frac{t_3}{3},\dots,\frac{t_{j+\ell-2k+2}}{j+\ell-2k+2}\right), \quad (6)$$

$$B_{j,k}(s_1+t_1,s_2+t_2,\dots,s_{j-k+1}+t_{j-k+1})=\sum_{\kappa=0}^{k}\sum_{\nu=0}^{j}\binom{j}{\nu}B_{\nu,\kappa}(s_1,s_2,\dots,s_{\nu-\kappa+1})B_{j-\nu,k-\kappa}(t_1,t_2,\dots,t_{j-\nu-k+\kappa+1}), \quad (7)$$

for $j\ge k\in\mathbb{N}_0$ can be found in [2, pp. 135-136] and [4]. The quantity on the right hand side of the formula (2) is called the Lah number in combinatorics.

The identity (16) in [5, Remark 5] and the identity (2.16) in [6, Remark 5] are seemingly wrong and conflict to the identity (4). The identity (7) can be equivalently written as

$$B_{j,k}(s_1+t_1,s_2+t_2,\dots,s_{j-k+1}+t_{j-k+1})=\sum_{r+s=k}\sum_{\ell+m=j}\binom{j}{\ell}B_{\ell,r}(s_1,s_2,\dots,s_{\ell-r+1})B_{m,s}(t_1,t_2,\dots,t_{m-s+1}),$$

see [5, Example 54] and [6, Example 56].

In [7, Theorem 5.1] and [8, Section 3], the interesting formula

$$B_{j,k}(t,1,0,\dots,0)=\frac{1}{2^{j-k}}\frac{j!}{k!}\binom{k}{j-k}t^{2k-j}, \quad j\ge k\in\mathbb{N}_0$$

was derived. In [9, Remark 3.1], the formula

$$B_{j,k}\left(1,1-z,(1-z)(1-2z),\dots,\prod_{\ell=0}^{j-k}(1-\ell z)\right)=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{j-1}(\ell-qz) \quad (8)$$

for $j\ge k\in\mathbb{N}_0$ and $z\in\mathbb{C}$ was concluded. In [10, Theorems 2.1 and 4.1], the formula (8) was proved to be equivalent to

$$B_{j,k}(\langle z\rangle_1,\langle z\rangle_2,\dots,\langle z\rangle_{j-k+1})=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle z\ell\rangle_j \quad (9)$$

for $j\ge k\in\mathbb{N}_0$ and $z\in\mathbb{C}$. In [11, Remark 7], the formulas (8) and (9) were equivalently rewritten as

$$B_{j,k}\left(1,1-z,(1-z)(1-2z),\dots,\prod_{\ell=0}^{j-k}(1-\ell z)\right)=\begin{cases}(-1)^kz^j\dfrac{j!}{k!}\displaystyle\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{\ell/z}{j}, & z\ne0\\ S(j,k), & z=0\end{cases}$$

and

$$B_{j,k}(\langle z\rangle_1,\langle z\rangle_2,\dots,\langle z\rangle_{j-k+1})=(-1)^k\frac{j!}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{z\ell}{j}$$

for $j\ge k\in\mathbb{N}_0$ and $z\in\mathbb{C}$, where the Stirling numbers of the second kind $S(j,k)$ can be analytically generated [12] by

$$\left(\frac{e^z-1}{z}\right)^k=\sum_{j=0}^{\infty}\frac{S(j+k,k)}{\binom{j+k}{k}}\frac{z^j}{j!},\quad k\ge0.$$

In the proof of [13, Theorem 2.1], the author derived the formula

$$B_{j,k}(\langle z\rangle_1,\langle z\rangle_2,\dots,\langle z\rangle_{j-k+1})=\sum_{i=k}^{j}s(j,i)z^iS(i,k) \tag{10}$$

for $j\ge k\in\mathbb{N}_0$ and $z\in\mathbb{C}$, where the Stirling numbers of the first kind $s(j,i)$ can be generated [14, 15] by

$$\langle z\rangle_j=\sum_{i=0}^{j}s(j,i)z^i$$

for $z\in\mathbb{C}$ and $j\in\mathbb{N}_0$. Replacing α by $-\alpha$ in (10) and utilizing the relation

$$\langle -z\rangle_j=(-1)^j(z)_j$$

give

$$B_{j,k}((z)_1,(z)_2,\dots,(z)_{j-k+1})=\sum_{i=k}^{j}|s(j,i)|z^iS(i,k)$$

for $j\ge k\in\mathbb{N}_0$ and $z\in\mathbb{C}$. These results and many other things have been reviewed and surveyed in [13, 16].

In terms of the partial Bell polynomials $B_{j,k}$, the famous Faá di Bruno formula can be expressed [2, p. 139, Theorem C] as

$$\frac{d^j[f\circ h(t)]}{dt^j}=\sum_{k=0}^{j}f^{(k)}(h(t))B_{j,k}(h'(t),h''(t),\dots,h^{(j-k+1)}(t)) \tag{11}$$

for $j\in\mathbb{N}_0$, where f and h are j-time differentiable functions on an interval I. For the history and developments of the Faá di Bruno formula (11), please refer to the monograph [17] and closely related references therein.

In this paper, in view of the Faá di Bruno formula (11) and with the help of some identities for the partial Bell polynomials $B_{j,k}$ mentioned above, we differentiated three composite functions

$$e^{at^b},\quad \exp\left[-\frac{\lambda(t-\mu)^2}{2\mu^2t}\right],\quad (\operatorname{sech}t)^\alpha$$

for $a,b,\lambda,\mu,\alpha\in\mathbb{R}$ with $\mu\ne0$, established several derivative formulas for them, and generalized the formula (2) by

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},(\ell+1)!,(\ell+2)!,\dots,(j-k+1)!)=\binom{j-\ell k-1}{k-1}\frac{j!}{k!}$$

for $j\ge k\in\mathbb{N}_0$ and $0\le\ell\le j-k+1$.

2. DERIVATIVE FORMULAS OF THE FIRST EXPONENTIAL FUNCTION

At the website https://math.stackexchange.com/q/3887532, the j th derivative of the exponential function e^{at^m} was asked for. The first main result of this paper is the following derivative formulas of the exponential function e^{at^b} for $a,b\in\mathbb{R}$.

Theorem 1 Let $a,b\in\mathbb{R}$ and $j\in\mathbb{N}_0$. Then the j th derivative of the exponential function e^{at^b} can be computed by the explicit formula

$$\frac{d^je^{at^b}}{dt^j}=\frac{e^{at^b}}{t^j}\sum_{k=0}^{j}a^k\left[\sum_{i=k}^{j}s(j,i)b^iS(i,k)\right]t^{bk}, \tag{12}$$

where $s(j,i)$ and $S(i,k)$ represent the Stirling numbers of the first and second kinds respectively.

Proof. This is a slightly modified version of the answer by the author at the website https://math.stackexchange.com/q/4656607.

By the Faá di Bruno formula (11) and the identity (1), we obtain

$$\frac{d^je^{at^b}}{dt^j}=\sum_{k=0}^{j}e^{at^b}B_{j,k}(a\langle b\rangle_1t^{b-1},a\langle b\rangle_2t^{b-2},\dots,a\langle b\rangle_{j-k+1}t^{b-(j-k+1)})$$

$$=e^{at^b}\sum_{k=0}^{j}a^kt^{bk-j}B_{j,k}(\langle b\rangle_1,\langle b\rangle_2,\dots,\langle b\rangle_{j-k+1})$$

for $j\in\mathbb{N}_0$. Further, utilizing the identity (10) results in

$$\frac{d^je^{at^b}}{dt^j}=e^{at^b}\sum_{k=0}^{j}a^kt^{bk-j}\sum_{i=k}^{j}s(j,i)b^iS(i,k)$$

$$=\frac{e^{at^b}}{t^j}\sum_{k=0}^{j}a^k\left[\sum_{i=k}^{j}s(j,i)b^iS(i,k)\right]t^{bk}$$

for $j\in\mathbb{N}_0$. The derivative formula (12) is thus proved.

Corollary 1 For $j\in\mathbb{N}_0$ and $a\in\mathbb{R}$, we have

$$\frac{d^j e^{at}}{dt^j}=a^j e^{at}, \tag{13}$$

$$\frac{d^j e^{at^2}}{dt^j}=e^{at^2}\frac{j!}{(2t)^j}\sum_{k=0}^{j}\frac{a^k}{k!}\binom{k}{j-k}(2t)^{2k},$$

$$\frac{d^j e^{a\sqrt{t}}}{dt^j}=e^{a\sqrt{t}}\frac{(-1)^j}{2^j t^j}\sum_{k=0}^{j}(-1)^k a^k[2(j-k)-1]!!\binom{2j-k-1}{2(j-k)}t^{k/2},$$

$$\frac{d^j e^{a/t}}{dt^j}=e^{a/t}\frac{(-1)^j j!}{t^j}\sum_{k=0}^{j}\frac{a^k}{k!}\binom{j-1}{k-1}\frac{1}{t^k},$$

$$\frac{d^j e^{a/t^2}}{dt^j}=e^{a/t^2}\frac{(-1)^j j!}{t^j}\sum_{k=0}^{j}(-1)^k\frac{a^k}{k!}\Bigl[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{j+2\ell-1}{j}\Bigr]\frac{1}{t^{2k}}. \tag{14}$$

Proof. In [13, Theorem 3.1], Qi and his coauthors obtained

$$\sum_{i=k}^{j}s(j,i)1^i S(i,k)=\binom{0}{j-k},$$

$$\sum_{i=k}^{j}s(j,i)2^i S(i,k)=\frac{j!}{k!}\binom{k}{j-k}2^{2k-j},$$

$$\sum_{i=k}^{j}s(j,i)\left(\frac{1}{2}\right)^i S(i,k)=(-1)^{j+k}\frac{[2(j-k)-1]!!}{2^j}\binom{2j-k-1}{2(j-k)},$$

$$\sum_{i=k}^{j}s(j,i)(-2)^i S(i,k)=(-1)^{j+k}\frac{j!}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\binom{j+2\ell-1}{j},$$

and

$$\sum_{i=k}^{j}s(j,i)(-1)^i S(i,k)=(-1)^j\frac{j!}{k!}\binom{j-1}{k-1} \tag{15}$$

for $j\ge k\in\mathbb{N}_0$. Substituting these identities into the derivative formula (12) and simplifying result in those derivative formulas between (13) and (14).

3. DERIVATIVE FORMULAS OF THE SECOND EXPONENTIAL FUNCTION

At the websites https://math.stackexchange.com/q/4902189 and https://mathoverflow.net/q/469595, the jth derivative of the exponential function

$$f(t)=\exp\Bigl[-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\Bigr],$$

where λ and $\mu\neq 0$ are two real numbers, was asked for.

Theorem 2 For $j\in\mathbb{N}_0$, the jth derivative of the function $f(t)$ can be computed by any one of the derivative formulas

$$\begin{aligned}f^{(j)}(t)&=\frac{f(t)}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\lambda^k(-2t)^{j-k}\\&\quad\times\sum_{\ell=0}^{k}\binom{j}{\ell}\binom{j-k-1}{k-\ell-1}\frac{(j-\ell)!}{(k-\ell)!}\mu^{2(j-\ell)}(\mu^2-t^2)^\ell,\end{aligned} \tag{16}$$

$$f^{(j)}(t)=f(t)\frac{(-1)^j j!}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\frac{\lambda^{j-k}}{(j-k)!}2^k\mu^{2k}\sum_{\ell=0}^{k}\frac{(-\lambda)^\ell}{(2\ell)!!}\binom{k-1}{\ell-1}t^{2j-k-\ell}, \tag{17}$$

and

$$\begin{aligned}f^{(j)}(t)&=f(t)\frac{1}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\binom{j}{k}(-\lambda)^{j-k}2^k\mu^{2k}t^{2(j-k)}\\&\quad\times\sum_{\ell=0}^{k}\Bigl(-\frac{\lambda}{2}\Bigr)^\ell\Bigl[\sum_{i=\ell}^{k}s(k,i)(-1)^i S(i,\ell)\Bigr]t^{k-\ell},\end{aligned} \tag{18}$$

where $s(k,i)$ and $S(i,\ell)$ stand for the Stirling numbers of the first and second kinds.

Proof. We will supply several proofs of these three derivative formulas.

First Proof of the Derivative Formula (16)

This proof is a slightly modified version of the answer by the author at the website https://math.stackexchange.com/q/4903130.

It is easy to see that we can write

$$h(t)=\ln f(t)=-\frac{\lambda(t-\mu)^2}{2\mu^2 t}=\frac{\lambda}{2\mu^2}\Bigl(2\mu-t-\frac{\mu^2}{t}\Bigr). \tag{19}$$

Hence, its derivatives are

$$h'(t)=\frac{\lambda}{2\mu^2}\Bigl(\frac{\mu^2}{t^2}-1\Bigr)=\frac{\lambda}{2}\Bigl(\frac{1}{t^2}-\frac{1}{\mu^2}\Bigr) \tag{20}$$

and

$$h^{(k+1)}(t)=(-1)^k\frac{\lambda}{2}\frac{(k+1)!}{t^{k+2}},\quad k\in\mathbb{N}. \tag{21}$$

Making use of the identities (1) and (3), we arrive at

$$B_{j,k}(h'(t),h''(t),\dots,h^{(j-k+1)}(t))$$

$$=B_{j,k}(\frac{\lambda}{2}(\frac{1}{t^2}-\frac{1}{\mu^2}),-\frac{\lambda}{2}\frac{2!}{t^3},\dots,(-1)^{j-k}\frac{\lambda}{2}\frac{(j-k+1)!}{t^{j-k+2}})$$

$$=(-1)^{j+k}(\frac{\lambda}{2})^k B_{j,k}(\frac{1}{t^2}-\frac{1}{\mu^2},\frac{2!}{t^3},\dots,\frac{(j-k+1)!}{t^{j-k+2}})$$

$$=(-1)^{j+k}(\frac{\lambda}{2})^k \sum_{\ell=0}^{k}\binom{j}{\ell}(\frac{1}{t^2}-\frac{1}{\mu^2})^{\ell} B_{j-\ell,k-\ell}(0,\frac{2!}{t^3},\dots,\frac{(j-k+1)!}{t^{j-k+2}})$$

$$=(-1)^{j+k}(\frac{\lambda}{2})^k \sum_{\ell=0}^{k}\binom{j}{\ell}(\frac{1}{t^2}-\frac{1}{\mu^2})^{\ell} \frac{B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)}{t^{j+k-2\ell}}$$

$$=(-1)^{j+k}(\frac{\lambda}{2})^k \frac{1}{t^{j+k}}\sum_{\ell=0}^{k}\binom{j}{\ell}(1-\frac{t^2}{\mu^2})^{\ell} B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)$$

for $j\ge k\in\mathbb{N}_0$. By the Faá di Bruno formula (11), we obtain

$$f^{(j)}(t)=\sum_{k=0}^{j}e^{h(t)}B_{j,k}(h'(t),h''(t),\dots,h^{(j-k+1)}(t))$$

$$=f(t)\sum_{k=0}^{j}(-1)^{j+k}(\frac{\lambda}{2})^k\frac{1}{t^{j+k}}\sum_{\ell=0}^{k}\binom{j}{\ell}(1-\frac{t^2}{\mu^2})^{\ell}$$

$$\times B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)$$

$$=\frac{f(t)}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\lambda^k(-2t)^{j-k}\sum_{\ell=0}^{k}\binom{j}{\ell}\mu^{2(j-\ell)}(\mu^2-t^2)^{\ell}$$

$$\times B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)$$

$$-1cm=\frac{f(t)}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\lambda^k(-2t)^{j-k}\sum_{\ell=0}^{k}\binom{j}{\ell}\binom{j-k-1}{k-\ell-1}\frac{(j-\ell)!}{(k-\ell)!}\mu^{2(j-\ell)}(\mu^2-t^2)^{\ell}$$

for $j\in\mathbb{N}_0$, where the derivation

$$\begin{aligned}
&B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)\\
&=\frac{(j-\ell)!}{(j-k)!}B_{j-k,k-\ell}(\frac{2!}{2},\frac{3!}{3},\dots,\frac{(j+\ell-2k+2)!}{j+\ell-2k+2})\\
&=\frac{(j-\ell)!}{(j-k)!}B_{j-k,k-\ell}(1!,2!,\dots,(j+\ell-2k+1)!)\\
&=\frac{(j-\ell)!}{(j-k)!}\binom{j-k-1}{k-\ell-1}\frac{(j-k)!}{(k-\ell)!}\\
&=\binom{j-k-1}{k-\ell-1}\frac{(j-\ell)!}{(k-\ell)!}
\end{aligned}\tag{22}$$

for $k\ge\ell\in\mathbb{N}_0$ used the identity (4) and the formula (2) in sequence. The desired results (26) and (16) are thus proved.

Second Proof of the Derivative Formula (16)

This proof is a slightly revised version of the answer by the author at the website https://mathoverflow.net/q/469670.

Making use of the identity (6), we obtain

$$B_{j,k}(h'(t),h''(t),\dots,h^{(j-k+1)}(t))$$

$$=\frac{j!}{(j-k)!}\sum_{\ell=0}^{k}\frac{[h'(t)]^{\ell}}{\ell!}B_{j-k,k-\ell}(\frac{h''(t)}{2},\frac{h'''(t)}{3},\dots,\frac{h^{(j+\ell-2k+2)}(t)}{j+\ell-2k+2})$$

$$=\frac{j!}{(j-k)!}\sum_{\ell=0}^{k}\frac{1}{\ell!}[\frac{\lambda}{2}(\frac{1}{t^2}-\frac{1}{\mu^2})]^{\ell}$$

$$\times B_{j-k,k-\ell}(-\frac{1}{2}\frac{\lambda}{2}\frac{2!}{t^3},\frac{1}{3}\frac{\lambda}{2}\frac{3!}{t^4},\dots,\frac{(-1)^{j+\ell-2k+1}}{j+\ell-2k+2}\frac{\lambda}{2}\frac{(j+\ell-2k+2)!}{t^{j+\ell-2k+3}})$$

$$=\frac{j!}{(j-k)!}\sum_{\ell=0}^{k}\frac{1}{\ell!}[\frac{\lambda}{2}(\frac{1}{t^2}-\frac{1}{\mu^2})]^{\ell}(\frac{\lambda}{2})^{k-\ell}\frac{(-1)^{j-k}}{t^{j+k-2\ell}}$$

$$\times B_{j-k,k-\ell}(1!,2!,\dots,(j+\ell-2k+1)!)$$

$$=\frac{(-1)^{j-k}}{(j-k)!}\frac{j!}{t^{j+k}}(\frac{\lambda}{2})^k\sum_{\ell=0}^{k}\frac{1}{\ell!}(1-\frac{t^2}{\mu^2})^{\ell}\binom{j-k-1}{k-\ell-1}\frac{(j-k)!}{(k-\ell)!}$$

$$=\frac{(-1)^{j+k}}{t^{j+k}}\frac{j!}{k!}(\frac{\lambda}{2})^k\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{j-k-1}{k-\ell-1}(1-\frac{t^2}{\mu^2})^{\ell}$$

for $j\ge k\in\mathbb{N}_0$, where we used the identity (1) and the formula (2). Further, by the Faá di Bruno formula (11), we acquire

$$f^{(j)}(t)=\sum_{k=0}^{j}e^{h(t)}B_{j,k}(h'(t),h''(t),\dots,h^{(j-k+1)}(t))$$

$$=e^{h(t)}\sum_{k=0}^{j}\frac{(-1)^{j+k}}{t^{j+k}}\frac{j!}{k!}(\frac{\lambda}{2})^k\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{j-k-1}{k-\ell-1}(1-\frac{t^2}{\mu^2})^{\ell}$$

$$=f(t)\frac{j!}{t^{2j}}\sum_{k=0}^{j}\frac{(-1)^{j-k}}{k!}(\frac{\lambda}{2})^k t^{j-k}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{j-k-1}{k-\ell-1}(1-\frac{t^2}{\mu^2})^{\ell}$$

$$=\frac{f(t)}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\lambda^k(-2t)^{j-k}\sum_{\ell=0}^{k}\binom{j}{\ell}$$

$$\times\binom{j-k-1}{k-\ell-1}\frac{(j-\ell)!}{(k-\ell)!}\mu^{2(j-\ell)}(\mu^2-t^2)^{\ell}$$

for $j \in \mathbb{N}_0$. The derivative formula (16) is thus proved again.

First Proof of the Derivative Formula (17)

This proof is a slightly revised version of the answer by the author at the website https://math.stackexchange.com/q/4904046.

The function $h(t)$ defined in (19) can be rewritten as

$$h(t) = \frac{\lambda}{\mu} - \frac{\lambda}{2\mu^2} t - \frac{\lambda}{2}\frac{1}{t}.$$

Accordingly, the function $f(t)$ can be reformulated as

$$f(t) = \exp(\frac{\lambda}{\mu})\exp(-\frac{\lambda}{2\mu^2} t)\exp(-\frac{\lambda}{2}\frac{1}{t}). \tag{23}$$

By virtue of the Leibniz rule for differentiation, with the help of the Faá di Bruno formula (11), and in view of the identity (1) and the formula (2), we arrive at

$$f^{(j)}(t) = \exp(\frac{\lambda}{\mu})\sum_{k=0}^{j}\binom{j}{k}[\exp(-\frac{\lambda}{2\mu^2}t)]^{(j-k)}[\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(k)}$$

$$= \exp(\frac{\lambda}{\mu})\sum_{k=0}^{j}\binom{j}{k}\exp(-\frac{\lambda}{2\mu^2}t)(-\frac{\lambda}{2\mu^2})^{j-k}[\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(k)}$$

$$= \exp(\frac{\lambda}{\mu} - \frac{\lambda}{2\mu^2}t)\sum_{k=0}^{j}\binom{j}{k}(-\frac{\lambda}{2\mu^2})^{j-k}$$

$$\times\sum_{\ell=0}^{k}\exp(-\frac{\lambda}{2}\frac{1}{t})B_{k,\ell}(\frac{\lambda}{2}\frac{1!}{t^2}, -\frac{\lambda}{2}\frac{2!}{t^3}, \dots, (-1)^{k-\ell+2}\frac{\lambda}{2}\frac{(k-\ell+1)!}{t^{k-\ell+2}})$$

$$= \exp(\frac{\lambda}{\mu} - \frac{\lambda}{2\mu^2}t - \frac{\lambda}{2}\frac{1}{t})\sum_{k=0}^{j}\binom{j}{k}(-\frac{\lambda}{2\mu^2})^{j-k}$$

$$\times\sum_{\ell=0}^{k}(\frac{\lambda}{2})^{\ell}\frac{(-1)^{k+\ell}}{t^{k+\ell}}B_{k,\ell}(1!, 2!, \dots, (k-\ell+1)!)$$

$$= f(t)\sum_{k=0}^{j}\binom{j}{k}(-\frac{\lambda}{2\mu^2})^{j-k}\sum_{\ell=0}^{k}(\frac{\lambda}{2})^{\ell}\frac{(-1)^{k+\ell}}{t^{k+\ell}}\binom{k-1}{\ell-1}\frac{k!}{\ell!}$$

$$= f(t)\frac{(-1)^j j!}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\frac{\lambda^{j-k}}{(j-k)!}2^k\mu^{2k}\sum_{\ell=0}^{k}\frac{(-\lambda)^{\ell}}{(2\ell)!!}\binom{k-1}{\ell-1}t^{2j-k-\ell}$$

for $j \in \mathbb{N}_0$. The derivative formula (17) is thus proved.

Second Proof of the Derivative Formula (17)

This proof is a slightly revised version of the answer by the author at the website https://mathoverflow.net/q/469805.

In [18, 19] and [20, Theorem 2.2], the derivative formula

$$(e^{\pm 1/t})^{(j)} = (-1)^j e^{\pm 1/t}\sum_{k=0}^{j}(\pm 1)^k\binom{j-1}{k-1}\frac{j!}{k!}\frac{1}{t^{j+k}}, \quad j \in \mathbb{N}_0 \tag{24}$$

was derived alternatively; see also [21, 22] and the articles collected at the website https://qifeng618.wordpress.com/2018/05/10/some-papers-related-to-the-function-e1-x-and-the-lah-numbers/ https://qifeng618.wordpress.com/2018/05/10/some-papers-related-to-the-function-e1-x-and-the-lah-numbers/ (accessed on 26 April 2024). From (24), it follows that

$$(e^{\alpha/t})^{(j)} = (-1)^j e^{\alpha/t}\sum_{k=0}^{j}\alpha^k\binom{j-1}{k-1}\frac{j!}{k!}\frac{1}{t^{j+k}}, \quad j \in \mathbb{N}_0. \tag{25}$$

Taking $\alpha = -\frac{\lambda}{2}$ in (25) gives

$$[\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(k)} = (-1)^k\exp(-\frac{\lambda}{2}\frac{1}{t})\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}\binom{k-1}{\ell-1}\frac{k!}{\ell!}\frac{1}{t^{k+\ell}}, \quad k \in \mathbb{N}_0.$$

Therefore, in light of the Leibniz rule for differentiation, we obtain

$$f^{(j)}(t) = [\exp(\frac{\lambda}{\mu})\exp(-\frac{\lambda}{2\mu^2}t)\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(j)}$$

$$= \exp(\frac{\lambda}{\mu})\sum_{k=0}^{j}\binom{j}{k}[\exp(-\frac{\lambda}{2\mu^2}t)]^{(j-k)}[\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(k)}$$

$$= \exp(\frac{\lambda}{\mu})\sum_{k=0}^{j}\binom{j}{k}\exp(-\frac{\lambda}{2\mu^2}t)(-\frac{\lambda}{2\mu^2})^{j-k}$$

$$\times(-1)^k\exp(-\frac{\lambda}{2}\frac{1}{t})\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}\binom{k-1}{\ell-1}\frac{k!}{\ell!}\frac{1}{t^{k+\ell}}$$

$$= \exp(\frac{\lambda}{\mu} - \frac{\lambda}{2\mu^2}t - \frac{\lambda}{2}\frac{1}{t})\sum_{k=0}^{j}\binom{j}{k}(-\frac{\lambda}{2\mu^2})^{j-k}$$

$$\times(-1)^k\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}\binom{k-1}{\ell-1}\frac{k!}{\ell!}\frac{1}{t^{k+\ell}}$$

$$= f(t)\frac{(-1)^j j!}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\frac{\lambda^{j-k}}{(j-k)!}2^k\mu^{2k}\sum_{\ell=0}^{k}\frac{(-\lambda)^{\ell}}{(2\ell)!!}\binom{k-1}{\ell-1}t^{2j-k-\ell}$$

for $j \in \mathbb{N}_0$. The proof of the derivative formula (17) is proved again.

Proof of the Derivative Formula (18)

This is a slightly revised version of the answer announced at the website https://mathoverflow.net/q/469812 by the author.

Employing the expression (23), using the Leibnitz rule for differentiation, and utilizing the derivative formula (12) yield

$$f^{(j)}(t)=\exp(\frac{\lambda}{\mu})\sum_{k=0}^{j}\binom{j}{k}[\exp(-\frac{\lambda}{2\mu^2}t)]^{(j-k)}[\exp(-\frac{\lambda}{2}\frac{1}{t})]^{(k)}$$

$$=\exp(\frac{\lambda}{\mu}-\frac{\lambda}{2\mu^2}t-\frac{\lambda}{2}\frac{1}{t})\sum_{k=0}^{j}\binom{j}{k}(-\frac{\lambda}{2\mu^2})^{j-k}\frac{1}{t^k}$$

$$\times\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}[\sum_{i=\ell}^{k}s(k,i)(-1)^{i}S(i,\ell)]\frac{1}{t^{\ell}}$$

$$=f(t)\frac{1}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\binom{j}{k}(-\lambda)^{j-k}2^k\mu^{2k}t^{2(j-k)}$$

$$\times\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}[\sum_{i=\ell}^{k}s(k,i)(-1)^{i}S(i,\ell)]t^{k-\ell}$$

for $j\in\mathbb{N}_0$. The derivative formula (18) is thus proved.

Third Proof of the Derivative Formula (17)

Taking the identity (15) into the formula (18) gives

$$f^{(j)}(t)=f(t)\frac{1}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\binom{j}{k}(-\lambda)^{j-k}2^k\mu^{2k}t^{2(j-k)}$$

$$\times\sum_{\ell=0}^{k}(-\frac{\lambda}{2})^{\ell}(-1)^{k}\frac{k!}{\ell!}\binom{k-1}{\ell-1}t^{k-\ell}$$

$$=f(t)\frac{(-1)^j j!}{2^j\mu^{2j}t^{2j}}\sum_{k=0}^{j}\frac{\lambda^{j-k}}{(j-k)!}2^k\mu^{2k}\sum_{\ell=0}^{k}\frac{(-\lambda)^{\ell}}{(2\ell)!!}\binom{k-1}{\ell-1}t^{2j-k-\ell}$$

for $j\in\mathbb{N}_0$. The third proof of the derivative formula (17) is complete.

Fourth Proof of the Derivative Formula (17)

By virtue of the Faá di Bruno formula (11), in view of the derivatives in (20) and (21), and with assistance of the identity (1), we derive

$$\frac{d^j}{dt^j}f(t)=f(t)\sum_{k=1}^{j}B_{j,k}(\frac{\lambda}{2}(\frac{1}{t^2}-\frac{1}{\mu^2}),-\frac{\lambda}{2}\frac{2!}{t^3},\frac{\lambda}{2}\frac{3!}{t^4},\dots,(-1)^{j-k}\frac{\lambda}{2}\frac{(j-k+1)!}{t^{j-k+2}})$$

$$=f(t)\sum_{k=1}^{j}\frac{(-1)^{j+k}}{t^{j+k}}(\frac{\lambda}{2})^{k}B_{j,k}(1-\frac{t^2}{\mu^2},2!,3!,\dots,(j-k+1)!)$$

for $j\in\mathbb{N}$. Taking $s_\ell=\ell!$ for $\ell\in\mathbb{N}$, $t_1=-\frac{t^2}{\mu^2}$, and $t_\ell=0$ for $\ell\ge 2$ in (7) yields

$$B_{j,k}(1-\frac{t^2}{\mu^2},2!,3!,\dots,(j-k+1)!)$$

$$=\sum_{\kappa=0}^{k}\sum_{\nu=0}^{j}\binom{j}{\nu}B_{\nu,\kappa}(1!,2!,\dots,(\nu-\kappa+1)!)B_{j-\nu,k-\kappa}(-\frac{t^2}{\mu^2},\underbrace{0,\dots,0}_{j-\nu-k+\kappa})$$

$$=\sum_{\kappa=0}^{k}\sum_{\nu=0}^{j}\binom{j}{\nu}\binom{\nu-1}{\kappa-1}\frac{\nu!}{\kappa!}B_{j-\nu,k-\kappa}(-\frac{t^2}{\mu^2},\underbrace{0,\dots,0}_{j-\nu-k+\kappa})$$

$$=\sum_{\kappa=0}^{k}\sum_{\nu=0}^{j}\binom{j}{\nu}\binom{\nu-1}{\kappa-1}\frac{\nu!}{\kappa!}(-\frac{t^2}{\mu^2})^{k-\kappa},\quad j-\nu=k-\kappa$$

where we used the formulas (2) and (5).

The proof of Theorem 2 is complete.

Corollary 2 The partial Bell polynomials $B_{j,k}$ for $j\ge k\in\mathbb{N}_0$ satisfy

$$B_{j,k}(0,2!,\dots,(j-k+1)!)=\binom{j-k-1}{k-1}\frac{j!}{k!}. \tag{26}$$

Proof. This is a rearrangement of the identity

$$B_{j-\ell,k-\ell}(0,2!,\dots,(j-k+1)!)=\binom{j-k-1}{k-\ell-1}\frac{(j-\ell)!}{(k-\ell)!}$$

deduced from (22).

4. GENERALIZATION OF A FORMULA FOR SPECIFIC PARTIAL BELL POLYNOMIALS

Comparing (2) with (26) motivates us to pose a problem: What is the general and explicit formula of the quantity

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},(\ell+1)!,(\ell+2)!,\dots,(j-k+1)!)$$

for $j\ge k\in\mathbb{N}_0$ and $1\le\ell\le j-k+1$?

Theorem 3 For $j\ge k\in\mathbb{N}_0$ and $0\le\ell\le j-k+1$, the partial Bell polynomials $B_{j,k}$ satisfy the identity

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},t_{\ell+1},t_{\ell+2},\dots,t_{j-k+1})=\frac{j!}{(j-\ell k)!}B_{j-\ell k,k}\Big(\frac{1!}{(\ell+1)!}t_{\ell+1},$$

$$\frac{2!}{(\ell+2)!}t_{\ell+2},\dots,\frac{[j-(\ell+1)k+1]!}{[j-(\ell+1)k+\ell+1]!}t_{j-(\ell+1)k+\ell+1}\Big), \tag{27}$$

where we use the convention that $B_{j-\ell k,k}=0$ for $(\ell+1)k>j\in\mathbb{N}_0$. Consequently, the formula

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},(\ell+1)!,(\ell+2)!,\dots,(j-k+1)!)=\binom{j-\ell k-1}{k-1}\frac{j!}{k!} \tag{28}$$

is valid for $j\ge k\in\mathbb{N}_0$ and $0\le\ell\le j-k+1$.

Proof. The identity (4) can be reformulated as

$$B_{j,k}(0,t_2,t_3,\dots,t_{j-k+1})=\frac{j!}{(j-k)!}B_{j-k,k}\Big(\frac{t_2}{2},\frac{t_3}{3},\dots,\frac{t_{j-2k+2}}{j-2k+2}\Big) \tag{29}$$

for $j\ge k\in\mathbb{N}_0$. Taking $t_2=0$ in (29) and applying (29) give

$$B_{j,k}(0,0,t_3,\dots,t_{j-k+1})=\frac{j!}{(j-k)!}B_{j-k,k}\Big(0,\frac{t_3}{3},\dots,\frac{t_{j-2k+2}}{j-2k+2}\Big)$$

$$=\frac{j!}{(j-k)!}\frac{(j-k)!}{(j-2k)!}B_{j-2k,k}\Big(\frac{t_3}{2\cdot3},\frac{t_4}{3\cdot4},\dots,\frac{t_{j-3k+3}}{(j-3k+2)(j-3k+3)}\Big)$$

$$=\frac{j!}{(j-2k)!}B_{j-2k,k}\Big(\frac{t_3}{2\cdot3},\frac{t_4}{3\cdot4},\dots,\frac{t_{j-3k+3}}{(j-3k+2)(j-3k+3)}\Big)$$

for $j\ge k\in\mathbb{N}_0$. Inductively, we conclude that

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},t_{\ell+1},t_{\ell+2},\dots,t_{j-k+1})$$

$$=\frac{j!}{(j-\ell k)!}B_{j-\ell k,k}\Big(\frac{t_{\ell+1}}{2\cdot3\cdots(\ell+1)},\frac{t_{\ell+2}}{3\cdot4\cdots(\ell+2)},\dots,$$
$$\frac{t_{j-(\ell+1)k+\ell+1}}{[j-(\ell+1)k+2][j-(\ell+1)k+3]\cdots[j-(\ell+1)k+\ell+1]}\Big)$$
$$=\frac{j!}{(j-\ell k)!}B_{j-\ell k,k}\Big(\frac{1!t_{\ell+1}}{(\ell+1)!},\frac{2!t_{\ell+2}}{(\ell+2)!},\dots,\frac{[j-(\ell+1)k+1]!t_{j-(\ell+1)k+\ell+1}}{[j-(\ell+1)k+\ell+1]!}\Big)$$

for $j\ge k\in\mathbb{N}_0$ and $1\le\ell\le j-k+1$. The identity (27) is thus proved.

Taking $t_\ell=\ell!$ in (27), simplifying, and employing the formula (2) lead to

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},(\ell+1)!,(\ell+2)!,\dots,(j-k+1)!)$$

$$=\frac{j!}{(j-\ell k)!}B_{j-\ell k,k}(1!,2!,\dots,[j-(\ell+1)k+1]!)$$

$$=\frac{j!}{(j-\ell k)!}\binom{j-\ell k-1}{k-1}\frac{(j-\ell k)!}{k!}$$

$$=\binom{j-\ell k-1}{k-1}\frac{j!}{k!}$$

for $j\ge k\in\mathbb{N}_0$ and $1\le\ell\le j-k+1$. The formula (28) is thus proved.

Remark 1 Letting $\ell=0,1$ in the formula (28) reduces to (2) and (26). So we can say that the formula (28) is a generalization of the formulas (2) and (26).

Remark 2 The formula (27) is a recovery of Excercise 7 in [4, p. 450].

5. DERIVATIVE FORMULA FOR REAL POWER OF HYPERBOLIC SECANT FUNCTION

At the website https://math.stackexchange.com/q/4219124, a clean formula for $\frac{d^j}{dt^j}\operatorname{sech}^2 t$ was asked for.

We now recite an answer by the author at the website https://math.stackexchange.com/q/4249446 as follows.

Theorem 4 For $\alpha\in\mathbb{R}$ and $j\in\mathbb{N}_0$, we have

$$\frac{d^j[(\operatorname{sech}t)^\alpha]}{dt^j}=\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}\sum_{\ell=0}^{k}\langle-\alpha\rangle_\ell 2^{k-\ell}S(k,\ell)(\sinh t+\cosh t)^\ell\operatorname{sech}^{\alpha+\ell}t, \tag{30}$$

where $S(k,\ell)$ stands for the Stirling numbers of the second kind and $\langle-\alpha\rangle_\ell$ is the falling factorial.

Proof. This is a slightly modified version of the answer posted at https://math.stackexchange.com/q/4249446 by the author.

The hyperbolic secant function $\operatorname{sech}t$ is defined [23, p. 83] by

$$\operatorname{sech}t=\frac{2}{e^t+e^{-t}}=\frac{2e^t}{1+e^{2t}}.$$

Then, when setting $h=h(t)=1+e^{2t}$, by virtue of the Leibnitz rule for differentiation and the Faá di Bruno formula (11), we obtain

$$\frac{d^j[(\operatorname{sech}t)^\alpha]}{dt^j}=2^\alpha\frac{d^j}{dt^j}\frac{e^{\alpha t}}{(1+e^{2t})^\alpha}$$

$$=2^\alpha\sum_{k=0}^{j}\binom{j}{k}(e^{\alpha t})^{(j-k)}[(1+e^{2t})^{-\alpha}]^{(k)}$$

$$=2^{\alpha}\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}e^{\alpha t}\sum_{\ell=0}^{k}(h^{-\alpha})^{(\ell)}B_{k,\ell}(h'(t),h''(t),\dots,h^{(k-\ell+1)}(t))$$

$$=2^{\alpha}\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}e^{\alpha t}\sum_{\ell=0}^{k}\frac{\langle-\alpha\rangle_{\ell}}{h^{\alpha+\ell}}B_{k,\ell}(2e^{2t},2^{2}e^{2t},\dots,2^{(k-\ell+1)}e^{2t})$$

$$=2^{\alpha}\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}e^{\alpha t}\sum_{\ell=0}^{k}\frac{\langle-\alpha\rangle_{\ell}}{(1+e^{2t})^{\alpha+\ell}}2^{k}e^{2\ell t}B_{k,\ell}(1,1,\dots,1)$$

$$=\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}\sum_{\ell=0}^{k}S(k,\ell)\langle-\alpha\rangle_{\ell}\frac{2^{\alpha+k}e^{(\alpha+2\ell)t}}{(1+e^{2t})^{\alpha+\ell}}$$

$$=\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}\sum_{\ell=0}^{k}S(k,\ell)\langle-\alpha\rangle_{\ell}2^{k-\ell}(\sinh t+\cosh t)^{\ell}\operatorname{sech}^{\alpha+\ell}t$$

for $\alpha\in\mathbb{R}$. The derivative formula (30) is thus proved.

Remark 3 Taking $\alpha=2$ in the formula (30) yields the derivative formula

$$\frac{d^{j}[(\operatorname{sech}t)^{2}]}{dt^{j}}=\sum_{k=0}^{j}\binom{j}{k}\sum_{\ell=0}^{k}(-1)^{\ell}(\ell+1)!2^{j-\ell}S(k,\ell)(\sinh t+\cosh t)^{\ell}\operatorname{sech}^{\ell+2}t$$

for $j\in\mathbb{N}_0$. The problem at the website https://math.stackexchange.com/q/4219124 is thus solved.

Remark 4 In [24, 25, 26], among other things, the nth derivative formula of the function $\operatorname{sech}t$ was alternatively computed, which is a special case $\alpha=1$ of the formula (30).

6. CONCLUSIONS

In this paper, in view of the Faá di Bruno formula (11) and with the help of some identities for the partial Bell polynomials $B_{j,k}$, we differentiated three composite functions

$$e^{at^{b}},\quad \exp[-\frac{\lambda(t-\mu)^{2}}{2\mu^{2}t}],\quad (\operatorname{sech}t)^{\alpha}$$

for $a,b,\lambda,\mu,\alpha\in\mathbb{R}$ with $\mu\ne0$, established several derivative formulas for them, and generalized the formula (2) by

$$B_{j,k}(\underbrace{0,\dots,0}_{\ell},(\ell+1)!,(\ell+2)!,\dots,(j-k+1)!)=\binom{j-\ell k-1}{k-1}\frac{j!}{k!}$$

for $j\ge k\in\mathbb{N}_0$ and $0\le\ell\le j-k+1$.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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