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Relation between Total Geodesic Submanifolds and Tachibana Operator for Lorentzian Sasakian Space Forms

Beyda Nur Altay¹, Tuğba Mert^{1,*} and Mehmet Atçeken²

¹Department of Mathematics, Faculty of Science, University of Sivas Cumhuriyet, 58140, Sivas, Turkey

²Department of Mathematics, Faculty of Art and Science, University of Aksaray, 68100, Aksaray, Turkey

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*Corresponding Author
E-mail: tmert@cumhuriyet.edu.tr

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Abstract:

In this article, total geodesic submanifolds of a Lorentzian Sasakian Space forms are characterized according to the Tachibana tensor with the help of some special curvature tensors. The behavior of total geodesic submanifolds of the Lorentzian Sasakian Space Forms has determined under some important conditions.

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1. INTRODUCTION

φ –sectional curvature plays the an important role for Sasakian manifold. If the φ –sectional curvature of a Sasakian manifold is constant, then the manifold is a Sasakian space form [1]. P. Alegre and D. Blair described generalized Sasakian space forms [2]. P. Alegre and D. Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. P. Alegre and A. Carriazo later discussed generalized indefinite Sasakian space forms [3]. Generalized indefinite Sasakian space forms are also called Lorentzian Sasakian space forms, and Lorentz manifolds are of great importance for Einstein’s theory of Relativity. Sasakian space forms, generalized Sasakian space forms and Lorentzian Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained [4-8].

Many mathematicians have considered the submanifolds of manifolds such as K -paracontact, Lorentzian para-Kenmotsu, almost Kenmotsu and studied their various characterizations [9-11].

On the other hand, the Tachibana operator is an important tool used in the field of differential geometry, especially when examining Sasakian manifolds. This operator is used to define and investigate some geometric properties of tensor fields on a manifold.

The Tachibana operator is a powerful tool in various areas of differential geometry and Riemannian geometry. This operator is used to understand the complex structure and curvature properties of manifolds, which contributes to understanding the general geometric and topological properties of manifolds.

The Tachibana operator plays an important role in differential geometry research and is used in various fields such as Sasakian manifolds, Einstein manifolds and generalized geometric structures.

In this article, total geodesic submanifolds of a Lorentzian Sasakian Space forms are characterized according to the Tachibana tensor with the help of some special curvature tensors. The behavior of total geodesic submanifolds of the Lorentzian Sasakian Space Forms has determined under some important conditions.

2. PRELIMINARIES

Let \tilde{M} be a $(2n + 1)$ – dimensional Lorentz manifold. If the \tilde{M} Lorentz manifold with (φ, ξ, η, g) structure

tensors satisfies the following conditions, this manifold is called a Lorentzian Sasakian manifold

$$\begin{aligned} \varphi^2 X &= -X + \eta(X)\xi, \eta(\xi) = 1, \eta(\varphi X) = 0, \\ g(\varphi X, \varphi Y) &= g(X, Y) + \eta(X)\eta(Y), \eta(X) = -g(X, \xi), \\ (\tilde{\nabla}_X \varphi)Y &= -g(X, Y)\xi - \eta(Y)X, \tilde{\nabla}_X \xi = -\varphi X, \end{aligned}$$

where, $\tilde{\nabla}$ is the Levi-Civita connection according to the Riemann metric g .

The plane section Π in $T_x \tilde{M}$. If the Π plane is spanned by X and φX , this plane is called the φ -section. The curvature of the φ -section is called the φ -sectional curvature. If the Lorentzian Sasakian manifold has a constant φ -sectional curvature, this manifold is called the Lorentzian Sasakian space form and is denoted by $\tilde{M}(c)$. The curvature tensor of the Lorentzian Sasakian space form $\tilde{M}(c)$ is defined as

$$\begin{aligned} \tilde{R}(X, Y)Z &= \left(\frac{c-3}{4}\right)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \left(\frac{c+1}{4}\right)\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\ &+ 2g(X, \varphi Y)\varphi Z + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \quad (1) \end{aligned}$$

for all $X, Y, Z \in \chi(\tilde{M})$.

Lemma 1 Let $\tilde{M}(c)$ be the $(2n + 1)$ – dimensional Lorentzian Sasakian space form. The following relations are provided for the Lorentzian Sasakian space forms.

$$\tilde{\nabla}_X \xi = -\varphi X, \tag{2}$$

$$(\tilde{\nabla}_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X, \tag{3}$$

$$(\tilde{\nabla}_X \eta)Y = g(\varphi X, Y), \tag{4}$$

$$\tilde{R}(\xi, Y)Z = -g(Y, Z)\xi - \eta(Z)Y, \tag{5}$$

$$\tilde{R}(\xi, Y)\xi = \eta(Y)\xi - Y, \tag{6}$$

$$\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{7}$$

$$S(X, \xi) = -\left[\frac{(c+1)-4m}{2}\right]\eta(X), \tag{8}$$

where \tilde{R}, S and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $\tilde{M}(c)$, respectively.

Let M be the immersed submanifold of the $(2n + 1)$ – dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. Let

the tangent and normal subspaces of M in $\tilde{M}(c)$ be $\Gamma(TM)$ and $\Gamma(T^\perp M)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma(T^\perp M)$ are

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{9}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \tag{10}$$

respectively, for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where ∇ and ∇^\perp are the connections on M and $\Gamma(T^\perp M)$, respectively, σ and A are the second fundamental form and the shape operator of M . There is a relation

$$g(A_V X, Y) = g(\sigma(X, Y), V) \tag{11}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$(\tilde{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{12}$$

Specifically, if $\tilde{\nabla} \sigma = 0$, M is said to be in the parallel second fundamental form or 1 –parallel.

Let R be the Riemann curvature tensor of M . In this case, the Gauss equation can be expressed as

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + A_{\sigma(X, Z)}Y - A_{\sigma(Y, Z)}X \\ &+ (\tilde{\nabla}_X \sigma)(Y, Z) - (\tilde{\nabla}_Y \sigma)(X, Z). \end{aligned} \tag{13}$$

Let M be a Riemannian manifold, T is $(0, k)$ –type tensor field and A is $(0, 2)$ –type tensor field. In this case, Tachibana tensor field $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(X_1, \dots, X_{k-1}; X, Y) &= -T((X \wedge_A Y)X, \dots, X_k) \\ &- T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \tag{14}$$

where,

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \tag{15}$$

$$k \geq 1, X_1, X_2, \dots, X_k, X, Y \in \Gamma(TM).$$

Let M be the submanifold of the Lorentzian Sasakian space form $\tilde{M}(c)$ and σ be the second fundamental form. In this case,

$$\begin{aligned} (\tilde{R}(X, Y) \cdot \sigma)(Z, W) &= R^\perp(X, Y)\sigma(Z, W) - \sigma(R(X, Y)Z, W) \\ &- \sigma(Z, R(X, Y)W). \end{aligned} \tag{16}$$

3. TOTAL GEODESIC INVARIANT SUBMANIFOLDS OF LORENTZIAN SASAKIAN SPACE FORMS

Let M be the immersed submanifold of a $(2n + 1)$ – dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. If

$\varphi(T_x M) \subset T_x M$ in every x point, the M manifold is called invariant submanifold. From this section of the article, we will assume that the manifold M is the invariant submanifold of the Lorentzian Sasakian space form $\tilde{M}(c)$. So it is clear from (3) and (9) that

$$\tilde{R}(X, Y)\xi = R(X, Y)\xi, \tag{17}$$

$$\sigma(\varphi X, Y) = \sigma(X, \varphi Y) = \varphi\sigma(X, Y), \tag{18}$$

$$\sigma(X, \xi) = 0 \tag{19}$$

for all $X, Y \in \Gamma(TM)$.

Now let us examine the $Q(g, \sigma) = 0$ and $Q(S, \sigma) = 0$ conditions for the $(2n + 1)$ –dimensional invariant submanifold M of a Lorentzian Sasakian space form $\tilde{M}(c)$.

Theorem 1 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(g, \sigma) = 0$, then M is a totally geodesic submanifold.

Proof. Let us assume that $Q(g, \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ –dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$\sigma(g(Y, U)X - g(X, U)Y, V) + \sigma(U, g(Y, V)X - g(X, V)Y) = 0, \tag{20}$$

for all $X, Y, U, V \in \Gamma(TM)$. If we choose $Y = U = \xi$ in (20) and use (19), then we have

$$\sigma(X, V) = 0.$$

This completes the proof.

Theorem 2 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(S, \sigma) = 0$, then M is either a totally geodesic submanifold or a space form with constant sectional curvature $c = 4n - 1$.

Proof. Let us assume that $Q(S, \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ –dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$\begin{aligned} \sigma(S(Y, U)X - S(X, U)Y, V) + \sigma(U, S(Y, V)X \\ - S(X, V)Y) = 0, \end{aligned} \tag{21}$$

for all $X, Y, U, V \in \Gamma(TM)$. If we choose $X = Y = \xi$ in (21) and use (8), (19), then we have

$$\left[\frac{(c + 1) - 4n}{2} \right] \sigma(U, Y) = 0.$$

This completes the proof.

Now let us examine the $Q(g, \tilde{\nabla} \cdot \sigma) = 0$ and $Q(S, \tilde{\nabla} \cdot \sigma) = 0$ condition for the $(2n + 1)$ -dimensional invariant submanifold M of a Lorentzian Sasakian space form $\tilde{M}(c)$.

Theorem 3 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(g, \tilde{\nabla} \cdot \sigma) = 0$, then M is a totally geodesic submanifold.

Proof. Let us assume that $Q(g, \tilde{\nabla} \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(\tilde{\nabla}_U \sigma)(g(Y, V)X - g(X, V)Y, Z) + (\tilde{\nabla}_U \sigma)(V, g(Y, Z)X - g(X, Z)Y) = 0, \tag{22}$$

for all $X, Y, Z, U, V, W \in \Gamma(TM)$. For $Y = V = \xi$ in (22), we can write

$$-(\tilde{\nabla}_U \sigma)(X, Z) + \eta(X)(\tilde{\nabla}_U \sigma)(\xi, Z) - \eta(Z)(\tilde{\nabla}_U \sigma)(\xi, X) = 0. \tag{23}$$

If we use (2), (12), (19), in (23), we can obtain

$$-\nabla_U^\perp \sigma(X, Z) + \sigma(\nabla_U X, Z) + \sigma(X, \nabla_U Z) + \eta(X)\varphi\sigma(U, Z) - \eta(Z)\varphi\sigma(U, X) = 0 \tag{24}$$

If we choose $Z = \xi$ and use (2), (19) in (24), we have

$$-2\varphi\sigma(X, U) = 0.$$

This completes the proof.

Theorem 4 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(S, \tilde{\nabla} \cdot \sigma) = 0$, then M is either a totally geodesic submanifold or a space form with constant sectional curvature $c = 4n - 1$.

Proof. Let us assume that $Q(S, \tilde{\nabla} \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(\tilde{\nabla}_U \sigma)(S(Y, V)X - S(X, V)Y, Z) + (\tilde{\nabla}_U \sigma)(V, S(Y, Z)X - S(X, Z)Y) = 0, \tag{25}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$. For $Y = Z = \xi$ in (25), this equality implies from (8),

$$-\frac{(c + 1) - 4n}{2} [\eta(V)(\tilde{\nabla}_U \sigma)(X, \xi) + (\tilde{\nabla}_U \sigma)(V, X) - \eta(X)(\tilde{\nabla}_U \sigma)(V, \xi)] = 0. \tag{26}$$

If we use (2), (12) in (26), we can obtain

$$-\frac{(c + 1) - 4n}{2} [\eta(V)\varphi\sigma(X, U) + \eta(X)\varphi\sigma(V, U) + \nabla_U^\perp \sigma(V, X) - \sigma(\nabla_U V, X) - \sigma(V, \nabla_U X)] = 0 \tag{27}$$

If we choose $V = \xi$ in (27), we have

$$\left[\frac{(c + 1) - 4n}{2} \right] \varphi\sigma(X, U) = 0.$$

This completes the proof.

Now let us examine the $Q(g, \tilde{R} \cdot \sigma) = 0$ and $Q(S, \tilde{R} \cdot \sigma) = 0$ conditions for the $(2n + 1)$ -dimensional invariant submanifold M of a Lorentzian Sasakian space form $\tilde{M}(c)$.

Theorem 5 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(g, \tilde{R} \cdot \sigma) = 0$, then M is a totally geodesic submanifold.

Proof. Let us assume that $Q(g, \tilde{R} \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(\tilde{R}(X, Y) \cdot \sigma)((U \wedge_g V)Z, W) + (\tilde{R}(X, Y) \cdot \sigma)(Z, (U \wedge_g V)W) = 0, \tag{28}$$

for all $X, Y, Z, U, V, W \in \Gamma(TM)$. For $Z = U = W = \xi$ in (28), this equality implies

$$(\tilde{R}(X, Y) \cdot \sigma)(-\eta(V)\xi + V, \xi) = 0. \tag{29}$$

If we use (7), (21) in (29), we get

$$\sigma(V, \eta(X)Y - \eta(Y)X) = 0. \tag{30}$$

If we choose $X = \xi$ in (30), we have

$$\sigma(V, Y) = 0.$$

This completes the proof.

Theorem 6 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(S, \tilde{R} \cdot \sigma) = 0$, then M is either a totally geodesic submanifold or space form with constant sectional curvature $c = 4n - 1$.

Proof. Let us assume that $Q(S, \tilde{R} \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(\tilde{R}(X, Y) \cdot \sigma)((U \wedge_S V)Z, W) + (\tilde{R}(X, Y) \cdot$$

$$\sigma)(Z, (U \wedge_S V)W) = 0, \tag{31}$$

for all $X, Y, Z, U, V, W \in \Gamma(TM)$. For $X = U = W = \xi$ in (31), this equality implies

$$(\tilde{R}(\xi, Y) \cdot \sigma)(S(V, Z)\xi, \xi) - S(\xi, Z)(\tilde{R}(\xi, Y) \cdot \sigma)(V, \xi) + S(V, \xi)(\tilde{R}(\xi, Y) \cdot \sigma)(Z, \xi) - S(\xi, \xi)(\tilde{R}(\xi, Y) \cdot \sigma)(Z, V) = 0. \tag{32}$$

If we choose $Z = \xi$ in (32), we get

$$S(\xi, \xi)\sigma(V, R(\xi, Y)\xi) = 0. \tag{38}$$

If we use (6), (12), (19) in (32), we obtain

$$[(c + 1) - 4n]\sigma(V, Y) = 0.$$

This completes the proof.

Now let us investigate the $Q(g, M \cdot \sigma) = 0$ and $Q(S, M \cdot \sigma) = 0$ conditions established with the M -projective curvature tensor for the M submanifold. To do this, let's first define the M -projective curvature tensor for $(2n + 1)$ -dimensional manifold M .

For a $(2n + 1)$ -dimensional Riemannian manifold (M, g) , the M -projective curvature tensor is defined as

$$M(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \tag{39}$$

for all $X, Y, Z \in \chi(M)$.

If we choose $Z = \xi$ in (39) for the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$, then we have

$$M(X, Y)\xi = \frac{(c+1)(n+3)}{8n}[\eta(Y)X - \eta(X)Y], \tag{40}$$

and similarly if we choose $X = \xi$ in (40), we have

$$M(\xi, Y)\xi = \frac{(c+1)(n+3)}{8n}[\eta(Y)\xi - Y]. \tag{41}$$

Theorem 7 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(g, M \cdot \sigma) = 0$, then M is either a totally geodesic submanifold or space form with constant sectional curvature $c = -1$.

Proof. Let us assume that $Q(g, M \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(M(X, Y) \cdot \sigma)((U \wedge_g V)Z, W) + (M(X, Y) \cdot \sigma)(Z, (U \wedge_g V)W) = 0, \tag{42}$$

for all $X, Y, Z, U, V, W \in \Gamma(TM)$. For $Z = U = W = \xi$ in (42), this equality implies

$$-(M(X, Y) \cdot \sigma)(\eta(V)\xi, \xi) + (M(X, Y) \cdot \sigma)(V, \xi) = 0, \tag{43}$$

and from (43)

$$\sigma(V, M(X, Y)\xi) = 0. \tag{44}$$

If we use (40) in (44), we can write

$$\frac{(c+1)(n+3)}{8n}\sigma(V, \eta(Y)X - \eta(X)Y) = 0. \tag{45}$$

If we choose $X = \xi$ in (45), we get

$$\frac{(c + 1)(n + 3)}{8n}\sigma(V, Y) = 0.$$

This completes the proof.

Theorem 8 Let M be an invariant submanifold of Lorentzian Sasakian space form $\tilde{M}(c)$. If $Q(S, M \cdot \sigma) = 0$, then at least one of the following is provided:

- i) M is totally geodesic submanifolds.
- ii) M is constant sectional curvature $c = 4n - 1$.
- iii) M is constant sectional curvature $c = -\frac{9n+1}{n+1}$.

Proof. Let us assume that $Q(S, M \cdot \sigma) = 0$ for the invariant submanifold M of the $(2n + 1)$ -dimensional Lorentzian Sasakian space form $\tilde{M}(c)$. In this case, we can write

$$(M(X, Y) \cdot \sigma)((U \wedge_S V)Z, W) + (M(X, Y) \cdot \sigma)(Z, (U \wedge_S V)W) = 0, \tag{46}$$

for all $X, Y, Z, U, V, W \in \Gamma(TM)$. For $X = U = W = \xi$ in (46), this equality implies

$$(M(\xi, Y) \cdot \sigma)(S(V, Z)\xi, \xi) - S(\xi, Z)(M(\xi, Y) \cdot \sigma)(V, \xi) + S(V, \xi)(M(\xi, Y) \cdot \sigma)(Z, \xi) - S(\xi, \xi)(M(\xi, Y) \cdot \sigma)(Z, V) = 0. \tag{47}$$

If we choose $Z = \xi$ in (47), we get

$$S(\xi, \xi)\sigma(M(\xi, Y)\xi, V) = 0. \tag{48}$$

If we use (8) and (40) in (48), we get

$$-\left[\frac{(c + 1) - 4n}{2}\right]\left[\frac{(n + 1)c + (9n + 1)}{8n}\right]\sigma(V, Y) = 0.$$

This completes the proof.

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