*Journal of Basic & Applied Sciences*, 2024, *20*, 203-210

Published by SET Publisher

**Journal of Basic & Applied Sciences**

ISSN (online): 1927-5129

# Intuitionistic Fuzzy Subalgebras of Several Types in Sheffer Stroke Hilbert Algebras

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**Article Info:**

*Keywords:*
Intuitionistic fuzzy set,
Intuitionistic fuzzy subalgebra.

*Timeline:*
Received: October 25, 2024
Accepted: November 22, 2024
Published: December 20, 2024

Citation: Rajesh N, Oner T, Senturk I, Ordin B. Intuitionistic Fuzzy Subalgebras of Several Types in Sheffer Stroke Hilbert Algebras. J Basic Appl Sci 2024; 20: 203-210.

*DOI:* https://doi.org/10.29169/1927-5129.2024.20.20

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**2020 Mathematics Subject Classification. 06F35, 03G25, 03B52.**

**Abstract:**

Characterizations of an $(\in,\in)$-intuitionistic fuzzy subalgebra and an $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$-intuitionistic fuzzy subalgebra are provided. Given special sets, so called intuitionistic fuzzy $\in$-subsets, intuitionistic fuzzy $q$-subsets and intuitionistic fuzzy $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$-subsets, conditions for the intuitionistic fuzzy $\in$-subsets, intuitionistic fuzzy $q$-subsets and intuitionistic fuzzy $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$-subsets to be subalgebras are discussed.

## 1. INTRODUCTION

The Sheffer stroke operation, also referred to as the NAND operator, was introduced by H. M. Sheffer [5]. This operation is notable for being functionally complete, meaning that any logical system can be constructed using only the Sheffer stroke [6]. Its functional completeness simplifies logical frameworks by eliminating the need for additional operators, thereby allowing for the reformulation of logical axioms and the axioms of Boolean algebra using a single operation. This characteristic highlights its versatility and fundamental role in both logical and algebraic systems.

The application of the Sheffer stroke in various algebraic structures has been widely studied, offering insights into diverse areas. Research has explored Sheffer stroke reducts of basic algebras [14, 19], Sheffer stroke MTL-algebras [18], ortholattices [2], and state operators in Sheffer stroke basic algebras [16]. Additionally, Riečan and Bosbach state operators on Sheffer stroke MTL-algebras [17] and fuzzy sets on Sheffer stroke BE-algebras [3] have been investigated, demonstrating the operation's wide-ranging implications.

The study of Sheffer stroke Hilbert-algebras has gained attention for their applications in non-classical logics and computational models, including intuitionistic fuzzy sets and quantum computing [8]. Previous work by Öner et al. [7, 9, 10, 15, 11, 12, 13] explored theoretical and axiomatic properties of these systems, uncovering connections between Sheffer stroke operations and innovative reasoning methods beyond classical Boolean logic.

This paper focuses on characterizing $(\in, \in)$-intuitionistic fuzzy subalgebras and $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$-intuitionistic fuzzy subalgebras. By examining special sets such as intuitionistic fuzzy $\in$-subsets, $q$ -subsets, and $(q_{(k_T,k_F)}, \in \vee\, q_{(k_T,k_F)})$-subsets, we establish conditions under which these subsets form subalgebras. For this aim, in Section 3, we define and analyze subsets derived from intuitionistic fuzzy sets, such as $T_\in$, $F_\in$, $T_q$, and $F_q$, based on membership and non-membership functions under specific thresholds. Theorems and proofs are provided to demonstrate how these subsets, under appropriate conditions, form intuitionistic fuzzy subalgebras, offering a robust foundation for further exploration of their algebraic and fuzzy properties.

## 2. PRELIMINARIES

The preliminaries section provides essential definitions, including Sheffer stroke operations, Sheffer stroke Hilbert algebras, and their key properties such as partial ordering and subalgebra preservation. Additionally, intuitionistic fuzzy sets are introduced to handle uncertainty, defined by membership and non-membership functions that enhance the algebraic structures' versatility.

**Definition 2.1** [5] Let $H = \langle H, | \rangle$ be a groupoid. The operation | is said to be a Sheffer stroke operation if it satisfies the following conditions:

$(S1)$ $x|y = y|x$

$(S2)$ $(x|x)|(x|y) = x$

$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$

$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

**Definition 2.2** [4] A Sheffer stroke Hilbert algebra is a structure $\langle H, | \rangle$ of type (2), in which $H$ is a non-empty set and | is a Sheffer stroke operation on $H$ such that the following identities are satisfied for all $x, y, z \in H$:

1. $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$,

2. If $x|(y|y) = y|(x|x) = x|(x|x)$ then $x = y$.

**Proposition 2.3** [4] Let $\langle A, | \rangle$ be a Sheffer stroke Hilbert algebra. Then the binary relation $x \le y$ if and only if $(x|(y|y)) = 1$ is a partial order on $A$.

**Definition 2.4** [4] A nonempty subset $G$ of a Sheffer stroke Hilbert algebra $X$ is called a subalgebra of $X$ if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.

An intuitionistic fuzzy set in $X$ (see [1]) is a structure of the form $A = \{\langle x, \mu_A(x), \psi_A(x) \rangle : x \in X\}$, where $\mu_A$: X→ $[0,1]$ is a membership function and $\psi_A: X \to [0,1]$ is a non-membership function. For the sake of simplicity, we use the symbol $A = (\mu_A, \psi_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \psi_A(x) \rangle : x \in X\}$.

## 3. INTUITIONISTIC FUZZY SUBALGEBRAS OF SEVERAL TYPES

In this section, we introduced the concept of intuitionistic fuzzy subalgebras of several types by defining subsets derived from intuitionistic fuzzy sets, such as $T_\in$, $F_\in$, $T_q$, and $F_q$. These subsets were analyzed based on membership and non-membership functions under specific conditions, such as $\alpha$ - and $\gamma$ - thresholds. Furthermore, we established several theorems and proofs demonstrating how these subsets, under appropriate conditions, form intuitionistic fuzzy subalgebras and explored their structural properties, providing a comprehensive foundation for the algebraic and fuzzy theoretical framework.

Given an intuitionistic fuzzy set $A = (A_T, A_F)$ in a set $X$, $\alpha \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T_\in(A, \alpha) = \{x \in X : A_T(x) \ge \alpha\},$$

$$F_{\in}(A,\gamma) = \{x \in X: A_F(x) \le \gamma\},$$

$$T_q(A,\alpha) = \{x \in X: A_T(x) + \alpha > 1\},$$

$$F_q(A,\gamma) = \{x \in X: A_F(x) + \gamma < 1\},$$

$$T_{\in\vee q}(A,\alpha) = \{x \in X: A_T(x) \ge \alpha \ or \ A_T(x) + \alpha > 1\},$$

$$F_{\in\vee q}(A,\gamma) = \{x \in X: A_F(x) \le \gamma \ or \ A_F(x) + \gamma < 1\}.$$

It is clear that

$$T_{\in\vee q}(A,\alpha) = T_{\in}(A,\alpha) \cup T_q(A,\alpha),$$

$$F_{\in\vee q}(A,\gamma) = F_{\in}(A,\gamma) \cup F_q(A,\gamma).$$

**Definition 3.1** Given $\Phi, \Psi \in \{\in, q, q \in\vee q\}$, an intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$ is called an $(\Phi,\Psi)$-intuitionistic fuzzy subalgebra of $X$ if the following assertions are valid:

$$\begin{gathered} x \in T_{\Phi}(A,\alpha_x), y \in T_{\Phi}(A,\alpha_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in T_{\Psi}(A, \alpha_x \wedge \alpha_y), \\ x \in F_{\Phi}(A,\gamma_x), y \in F_{\Phi}(A,\gamma_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in F_{\Psi}(A, \gamma_x \vee \gamma_y), \end{gathered} \tag{3.1}$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

**Theorem 3.2** An intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$ is an $(\in,\in)$-intuitionistic fuzzy subalgebra of $X$ if and only if

$$(\forall x, y \in X) \begin{pmatrix} A_T((x|(y|y))|(x|(y|y))) \ge A_T(x) \wedge A_T(y) \\ A_F((x|(y|y))|(x|(y|y))) \le A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.2}$$

**Theorem 3.3** If $A = (A_T, A_F)$ is an $(\in,\in)$-intuitionistic fuzzy subalgebra of $X$, then the subsets $T_{q_{k_T}}(A,\alpha)$ and $F_{q_{k_T}}(A,\gamma)$ are subalgebras of $X$ for all $\alpha \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.

Proof. Let $x, y \in T_{q_{k_T}}(A,\alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. It follows that $A_T((x|(y|y))|(x|(y|y))) + \alpha + k_T \ge (A_T(x) \wedge A_T(y)) + \alpha + k_T = (A_T(x) + \alpha + k_T) \wedge (A_T(y) + \alpha + k_T) > 1$ and so that $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A,\alpha)$. Hence $T_{q_{k_T}}(A,\alpha)$ is a subalgebra of $X$. Now let $x, y \in F_{q_{k_F}}(A,\beta)$. Then $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$, which imply that $A_F((x|(y|y))|(x|(y|y))) + \gamma + k_F \le (A_F(x) \vee A_F(y)) + \gamma + k_F = (A_F(x) + \gamma + k_F) \vee (A_F(y) + \gamma + k_F) < 1$. Hence $(x|(y|y))|(x|(y|y)) \in F_{q_{k_T}}(A,\gamma)$ and $F_{q_{k_T}}(A,\gamma)$ is a subalgebra of $X$.

**Theorem 3.4** If $A = (A_T, A_F)$ is a $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$-intuitionistic fuzzy subalgebra of $X$, then the subsets $T_{q_{k_T}}(A,\alpha)$ and $F_{q_{k_F}}(A,\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.

Proof. Let $x, y \in T_{q_{k_T}}(A,\alpha)$. Then $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q_{k_T}}(A,\alpha)$, and so $(x|(y|y))|(x|(y|y)) \in T_{\in}(A,\alpha)$ or $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A,\alpha)$. If $(x|(y|y))|(x|(y|y)) \in T_{\in}(A,\alpha)$, then $A_T((x|(y|y))|(x|(y|y))) \ge \alpha > 1 - \alpha$ since $\alpha > \frac{1-k_T}{2}$. Hence $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A,\alpha)$. Therefore $T_{q_{k_T}}(A,\alpha)$ is a subalgebra of $X$. Let $x, y \in F_{q_{k_F}}(A,\gamma)$. Then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$, and so $(x|(y|y))|(x|(y|y)) \in F_{\in}(A,\alpha)$ or $(x|(y|y))|(x|(y|y)) \in F_{q_{k_F}}(A,\alpha)$. If $(x|(y|y))|(x|(y|y)) \in F_{\in}(A,\gamma)$, then $A_F((x|(y|y))|(x|(y|y))) \le \gamma < \gamma - 1$ since $\gamma \in [\frac{1-k_F}{2}, 1)$. Hence $(x|(y|y))|(x|(y|y)) \in F_{q_{k_F}}(A,\gamma)$, and therefore $F_{q_{k_F}}(A,\gamma)$ is a subalgebra of $X$.

**Theorem 3.5** An intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$ is an $(\in, \in\vee\, q_{(k_T,k_F)})$-intuitionistic fuzzy subalgebra of $X$ if and only if

$$(\forall x, y \in X) \begin{pmatrix} A_T((x|(y|y))|(x|(y|y))) \ge \wedge \quad \{A_T(x) \wedge A_T(y), \frac{1-k_T}{2}\} \\ A_F((x|(y|y))|(x|(y|y))) \le \vee \quad \{A_F(x) \vee A_F(y), \frac{1-k_T}{2}\} \end{pmatrix}. \tag{3.3}$$

Proof. Suppose that $A = (A_T, A_F)$ is an $(\in, \in\vee\, q_{(k_T,k_F)})$-intuitionistic fuzzy subalgebra of $X$ and let $x, y \in X$. If $A_T(x) \wedge A_T(y) < \frac{1-k_T}{2}$, then we have $A_T((x|(y|y))|\ (x|(y|y))) \ge A_T(x) \wedge A_T(y)$. Suppose that $A_T((x|(y|y))|(x|(y|y))) < A_T(x) \wedge A_T(y)$ and choose $\alpha_t$ such that $A_T((x|(y|y))|(x|(y|y))) < \alpha_t < A_T(x) \wedge A_T(y)$. Then $x \in T_{\in}(A,\alpha_t)$ and $y \in T_{\in}(A,\alpha_t)$ but $(x|(y|y))|(x|(y|y)) \notin T_{\in}(A,\alpha_t)$. Also $A_T((x|(y|y))|(x|(y|y))) + \alpha_t < 1 - k_T$, that is, $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A,\alpha_t)$. Thus $(x|(y|y))|(x|(y|y)) \notin T_{\in\vee q_{k_T}}(A,\alpha_t)$, a contradiction. Then $A_T((x|(y|y))|(x|(y|y))) \ge \vee \quad \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ whenever $A_T(x) \wedge A_T(y) < \frac{1-k_T}{2}$. Now suppose that $A_T(x) \wedge A_T(y) \ge \frac{1-k_T}{2}$. Then $x \in T_{\in}(A, \frac{1-k_T}{2})$ and $y \in T_{\in}(A, \frac{1-k_T}{2})$, which imply that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q_{k_T}}(A, \frac{1-k_T}{2})$. Hence $A_T((x|(y|y))|(x|(y|y))) \ge \frac{1-k_T}{2}$. Otherwise, $A_T((x|(y|y))|(x|(y|y))) + \frac{1-k_T}{2} < \frac{1-k_T}{2} + \frac{1-k_T}{2} = 1 - k_T$, a contradiction. Consequently, $A_T((x|(y|y))|(x|(y|y))) \ge \vee \quad \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Suppose that $A_F(x) \vee A_F(y) > \frac{1-k_F}{2}$. If $A_F((x|(y|y))|(x|(y|y))) > A_F(x) \vee A_F(y) = \gamma$, then $x, y \in F_{\in}(A,\gamma)$, not $(x|(y|y))|(x|(y|y)) \notin F_{\in}(A,\gamma)$ and $A_F((x|(y|y))|(x|(y|y))) + \gamma > 2\gamma > 1 - k_F$, that is, $(x|(y|y))|(x|(y|y)) \notin F_{q_{k_F}}(A,\gamma)$, a contradiction. Hence $A_F((x|(y|y))|(x|(y|y))) \le \wedge \quad \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ whenever $A_F(x) \vee A_F(y) > \frac{1-k_F}{2}$. Now, assume that

$A_F(x) \vee A_F(y) \leq \frac{1-k_F}{2}$. Then $x, y \in F_\in(A, \frac{1-k_F}{2})$ and so $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q_{k_T}}(A, \frac{1-k_F}{2})$. Thus $A_F((x|(y|y))|(x|(y|y))) \leq \frac{1-k_F}{2}$ or $A_F((x|(y|y))|(x|(y|y))) + \frac{1-k_F}{2} < 1 - k_F$. If $A_F((x|(y|y))|(x|(y|y))) > \frac{1-k_F}{2}$, then $A_F((x|(y|y))|(x|(y|y))) + \frac{1-k_F}{2} > \frac{1-k_F}{2} + \frac{1-k_F}{2} = 1 - k_F$, a contradiction. Thus $A_F((x|(y|y))|(x|(y|y))) \leq \frac{1-k_F}{2}$, and so $A_F((x|(y|y))|(x|(y|y))) \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ whenever $A_F(x) \vee A_F(y) \leq \frac{1-k_F}{2}$. Therefore $A_F((x|(y|y))|(x|(y|y))) \leq \wedge \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Conversely, let $A = (A_T, A_F)$ be an intuitionistic fuzzy set in $X$ which satisfies the condition (3.3). Let $x, y \in X$ and $\alpha_x, \alpha_y, \gamma_x, \gamma_y \in [0,1]$. If $x \in T_\in(A, \alpha_x)$ and $y \in T_\in(A, \alpha_y)$, then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. If $A_T((x|(y|y))|(x|(y|y))) < \alpha_x \wedge \alpha_y$, then $A_T(x) \wedge A_T(y) \geq \frac{1-k_T}{2}$. Otherwise, we have $A_T((x|(y|y))|(x|(y|y))) \geq \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} = A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, a contradiction. It follows that $A_T((x|(y|y))|(x|(y|y))) + \alpha_x \wedge \alpha_y > 2A_T((x|(y|y))|(x|(y|y))) \geq 2 \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} = 1 - k_T$ and so that $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A, \alpha_x \wedge \alpha_y) \subseteq T_{\in\vee q_{k_T}}(A, \alpha_x \wedge \alpha_y)$. Now, let $x \in F_\in(A, \gamma_x)$ and $y \in F_\in(A, \gamma_y)$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $A_F((x|(y|y))|(x|(y|y))) > \gamma_x \vee \gamma_y$, then $A_F(x) \vee A_F(y) \leq \frac{1-k_F}{2}$ because if not, then $A_F((x|(y|y))|(x|(y|y))) \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y$; which is a contradiction. Hence $A_F((x|(y|y))|(x|(y|y))) + \gamma_x \vee \gamma_y < 2A_F((x|(y|y))|(x|(y|y))) \leq 2 \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} = 1 - k_F$; and so $(x|(y|y))|(x|(y|y)) \in F_{q_{k_F}}(A, \gamma_x \vee \gamma_y) \subseteq F_{\in\vee q_{k_F}}(A, \gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$.

**Theorem 3.6** If $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$, then the subsets $T_{q_{k_T}}(A, \alpha)$ and $F_{q_{k_T}}(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.

Proof. Assume that $T_{q_{k_T}}(A, \alpha)$ and $F_{q_{k_F}}(A, \gamma)$ are nonempty for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Let $x, y \in T_{q_{k_T}}(A, \alpha)$. Then $A_T(x) + \alpha + \frac{1-k_T}{2} > 1$ and $A_T(y) + \alpha + \frac{1-k_T}{2} > 1 - k_T$. By Theorem 3.5, $A_T((x|(y|y))|(x|(y|y))) + \alpha + \frac{1-k_T}{2} \geq \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} + \alpha = \wedge \{A_T(x) + \alpha, A_T(y) + \alpha, \frac{1-k_T}{2} + \alpha\} > 1 - k_T$; that is, $(x|(y|y))|(x|(y|y)) \in T_{q_{k_T}}(A, \alpha)$. Hence $T_{q_{k_T}}(A, \alpha)$ is a subalgebra of $X$. Now, let $x, y \in F_{q_{k_F}}(A, \gamma)$. Then $A_F(x) + \gamma + \frac{1-k_F}{2} < 1 - k_F$ and $A_F(y) + \gamma + \frac{1-k_F}{2} < 1 - k_F$. Using Theorem 3.5, we have $A_F((x|(y|y))|(x|(y|y))) + \gamma + \frac{1-k_F}{2} \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} + \gamma = \vee \{A_F(x) + \gamma, A_F(y) + \gamma, \frac{1-k_F}{2} + \gamma\} < 1 - k_F$; and so $(x|(y|y))|(x|(y|y)) \in F_{q_{k_F}}(A, \gamma)$. Hence $F_{q_{k_F}}(A, \gamma)$ is a subalgebra of $X$.

**Theorem 3.7** An intuitionistic fuzzy set $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$ if and only if the subsets $T_\in(A, \alpha)$ and $F_\in(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.

Proof. Let $A = (A_T, A_F)$ be an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$. For any $x, y \in X$, let $\alpha \in (0, \frac{1-k_T}{2}]$ be such that $x, y \in T_\in(A, \alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from 3.4 that $A_T((x|(y|y))|(x|(y|y))) \geq \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} \geq \alpha \wedge \frac{1-k_T}{2} = \alpha$ and so that $(x|(y|y))|(x|(y|y)) \in T_\in(A, \alpha)$. Hence $T_\in(A, \alpha)$ is a subalgebra of $X$. Now, let $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $x, y \in F_\in(A, \gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. It follows from 3.4 that $A_F((x|(y|y))|(x|(y|y))) \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \leq \gamma \vee \frac{1-k_F}{2} = \gamma$ and so that $(x|(y|y))|(x|(y|y)) \in F_\in(A, \gamma)$. Hence $F_\in(A, \alpha)$ is a subalgebra of $X$. Conversely, let $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $T_\in(A, \alpha)$ and $F_\in(A, \gamma)$ are subalgebras of $X$. If there exist $a, b \in X$ such that $A_T((a|(b|b))|(a|(b|b))) < \wedge \{A_T(a), A_T(b), \frac{1-k_T}{2}\}$, then we can take $\alpha \in (0,1)$ such that $A_T((a|(b|b))|(a|(b|b))) < \alpha < \wedge \{A_T(a), A_T(b), \frac{1-k_T}{2}\}$. Thus $a, b \in T_\in(A, \alpha)$ and $\alpha < \frac{1-k_T}{2}$, and so $(a|(b|b))|(a|(b|b)) \in T_\in(A, \alpha)$. But, the left inequality in induces $(a|(b|b))|(a|(b|b)) \notin T_\in(A, \alpha)$, a contradiction. Hence $A_T((x|(y|y))|(x|(y|y))) \geq \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now suppose that $A_F((a|(b|b))|(a|(b|b))) > \vee \{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $a, b \in X$. Then there exists $\gamma \in (0,1)$ such that $A_F((a|(b|b))|(a|(b|b))) > \gamma > \vee \{A_F(a), A_F(b), \frac{1-k_F}{2}\}$. It follows that $\gamma \in [\frac{1-k_F}{2}, 1)$ and $a, b \in F_\in(A, \gamma)$. Since $F_\in(A, \gamma)$ is a subalgebra of $X$, $(a|(b|b))|(a|(b|b)) \in F_\in(A, \gamma)$ and so $A_F((a|(b|b))|(a|(b|b))) \leq \gamma$. This is a contradiction, and thus $A_F((x|(y|y))|(x|(y|y))) \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Hence

$A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$.

Using Theorems 3.6 and 3.7, we have the following corollary.

**Corollary 3.8** For an intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$, if the nonempty subsets $T_{\in\vee q_{k_T}}(A, \alpha)$ and $F_{\in\vee q_{k_T}}(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then the subsets $T_\in(A, \alpha)$ and $F_\in(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

**Theorem 3.9** Given an intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$, the nonempty subsets $T_\in(A, \alpha)$ and $F_\in(A, \gamma)$ of $X$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ if and only if

$$(\forall x, y \in X)\begin{pmatrix} A_T((x|(y|y))|(x|(y|y))) \vee \frac{1-k_T}{2} \geq A_T(x) \wedge A_T(y) \\ A_F((x|(y|y))|(x|(y|y))) \wedge \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.4}$$

Proof. Assume that the nonempty subsets $T_\in(A, \alpha)$ and $F_\in(A, \gamma)$ of $X$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Suppose that there are $a, b \in X$ such that $A_T((a|(b|b))|(a|(b|b))) \vee \frac{1-k_T}{2} < A_T(a) \wedge A_T(b) = \alpha$. Then $\alpha \in (0, \frac{1-k_T}{2}]$ and $a, b \in T_\in(A, \alpha)$. Since $T_\in(A, \alpha)$ is a subalgebra of $X$, $(a|(b|b))|(a|(b|b)) \in T_\in(A, \alpha)$, that is, $A_T((a|(b|b))|(a|(b|b))) \geq \alpha$, a contradiction. Thus $A_T((x|(y|y))|(x|(y|y))) \vee \frac{1-k_T}{2} \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. Now, if $A_F((x|(y|y))|(x|(y|y))) \wedge \frac{1-k_F}{2} > A_F(x) \vee A_F(y)$ for some $x, y \in X$, then $x, y \in F_\in(A, \gamma)$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ where $\gamma = A_F(x) \vee A_F(y)$. But, $(x|(y|y))|(x|(y|y)) \notin F_\in(A, \gamma)$, a contradiction. Hence $A_F((x|(y|y))|(x|(y|y))) \wedge \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$. Conversely, let $A = (A_T, A_F)$ be an intuitionistic fuzzy set in $X$ satisfying the condition (3.4). Let $x, y \in X$ and $\alpha \in (0, \frac{1-k_T}{2}]$ be such that $x, y \in T_\in(A, \alpha)$. Then $A_T((x|(y|y))|(x|(y|y))) \vee \frac{1-k_T}{2} \geq A_T(x) \wedge A_T(y) \geq \alpha > \frac{1-k_T}{2}$. It follows that $A_T((x|(y|y))|(x|(y|y))) \geq \alpha$, that is, $(x|(y|y))|(x|(y|y)) \in T_\in(A, \alpha)$. Now, let $x, y \in X$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ be such that $x, y \in F_\in(A, \gamma)$. Then $A_F((x|(y|y))|(x|(y|y))) \vee \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y) \leq \gamma < \frac{1-k_F}{2}$ and so $A_F((x|(y|y))|(x|(y|y))) \leq \gamma$, that is, $(x|(y|y))|(x|(y|y)) \in F_\in(A, \gamma)$.

**Theorem 3.10** For an intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$, if the nonempty subsets $T_{\in\vee q}(A, \alpha)$ and $F_{\in\vee q}(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0,1]$ and $\gamma \in [0,1)$, then $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$.

Proof. Let $T_{\in\vee q}(A, \alpha)$ be a subalgebra of $X$ and $A_T((x|(y|y))|(x|(y|y))) < \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ for some $x, y \in X$. Then there exists $\alpha \in (0, \frac{1-k_T}{2}]$ such that $A_T((x|(y|y))|(x|(y|y))) < \alpha \leq \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$. It follows that $x, y \in T_\in(A, \alpha) \subseteq T_{\in\vee q}(A, \alpha)$, and so that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A, \alpha)$. Hence $A_T((x|(y|y))|(x|(y|y))) \geq \alpha$ or $A_T((x|(y|y))|(x|(y|y))) + \alpha > 1$, a contradiction, and so $A_T((x|(y|y))|(x|(y|y))) \geq \wedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Now let $F_{\in\vee q}(A, \gamma)$ be a subalgebra of $X$ and assume that $A_F((x|(y|y))|(x|(y|y))) > \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for some $x, y \in X$. Then

$$A_F((x|(y|y))|(x|(y|y))) > \gamma \geq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \tag{3.5}$$

for some $\gamma \in [\frac{1-k_F}{2}, 1)$, which implies that $x, y \in F_\in(A, \gamma) \subseteq F_{\in\vee q}(A, \gamma)$. Thus $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A, \gamma)$. From (3.5), $(x|(y|y))|(x|(y|y)) \notin F_\in(A, \gamma)$ and $A_F((x|(y|y))|(x|(y|y))) + \gamma > \in \geq 1$, that is, $(x|(y|y))|(x|(y|y)) \notin F_q(A, \gamma)$, a contradiction, and hence $A_F((x|(y|y))|(x|(y|y))) \leq \vee \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. By Theorem 3.5, $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$.

**Theorem 3.11** If $A = (A_T, A_F)$ is an $(\in, \in\vee q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$, then the nonempty subsets $T_{\in\vee q}(A, \alpha)$ and $F_{\in\vee q}(A, \gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

Proof. Assume that $T_{\in\vee q}(A, \alpha)$ and $F_{\in\vee q}(A, \gamma)$ are nonempty for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Let $x, y \in T_{\in\vee q}(A, \alpha)$. Then $x \in T_\in(A, \alpha)$ or $x \in T_q(A, \alpha)$, and $y \in T_\in(A, \alpha)$ or $y \in T_q(A, \alpha)$. Hence we have the following four cases:

1. $x \in T_\in(A, \alpha)$ and $y \in T_\in(A, \alpha)$,
2. $x \in T_\in(A, \alpha)$ and $y \in T_q(A, \alpha)$,
3. $x \in T_q(A, \alpha)$ and $y \in T_\in(A, \alpha)$,
4. $x \in T_q(A, \alpha)$ and $y \in T_q(A, \alpha)$.

The first case implies that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A, \alpha)$. For the second case, $y \in T_q(A, \alpha)$ induces

$A_T(y) > 1-\alpha \geq \alpha$, that is, $y \in T_\in(A,\alpha)$. Thus $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Similarly, the third case implies $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. The last case induces $A_T(x) > 1-\alpha \geq \alpha$ and $A_T(y) > 1-\alpha \geq \alpha$, that is, $x \in T_\in(A,\alpha)$ and $y \in T_\in(A,\alpha)$. Hence $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Therefore $T_{\in\vee q}(A,\alpha)$ is a subalgebra of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$. Let $x,y \in F_{\in\vee q}(A,\gamma)$. Then $A_F(x) \leq \gamma$ or $A_F(x)+\gamma < 1$, and $A_F(y) \leq \gamma$ or $A_F(y)+\gamma < 1$. If $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$, then $A_F((x|(y|y))|(x|(y|y))) \leq \vee\ \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \leq \vee\ \{\gamma, \frac{1-k_F}{2}\} = \gamma$ by Theorem 3.5, and so $(x|(y|y))|(x|(y|y)) \in F_\in(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. If $A_F(x) \leq \gamma$ and $A_F(y)+\gamma < 1$, then $A_F((x|(y|y))|(x|(y|y))) \leq \vee\ \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \leq \vee\ \{\gamma, 1-\gamma, \frac{1-k_F}{2}\} = \gamma$ by Theorem 3.5. Thus $(x|(y|y))|(x|(y|y)) \in F_\in(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. Similarly, if $A_F(x)+\gamma < 1$ and $A_F(y) \leq \gamma$, then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Finally, assume that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$. Then $A_F((x|(y|y))|(x|(y|y))) \leq \vee\ \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \leq \vee\ \{1-\gamma, \frac{1-k_F}{2}\} = \frac{1-k_F}{2} < \gamma$. Hence $(x|(y|y))|(x|(y|y)) \in F_\in(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. Consequently, $F_{\in\vee q}(A,\gamma)$ is a subalgebra of $X$ for all $\gamma \in [\frac{1-k_F}{2}, 1)$.

**Theorem 3.12** If $A = (A_T, A_F)$ is an $(q, \in\vee\, q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebra of $X$, then the nonempty subsets $T_{\in\vee q}(A,\alpha)$ and $F_{\in\vee q}(A,\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

Proof. Assume that $T_{\in\vee q}(A,\alpha)$ and $F_{\in\vee q}(A,\gamma)$ are nonempty for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Let $x,y \in T_{\in\vee q}(A,\alpha)$. Then $x \in T_\in(A,\alpha)$ or $x \in T_q(A,\alpha)$. and $y \in T_\in(A,\alpha)$ or $y \in T_q(A,\alpha)$. If $x \in T_q(A,\alpha)$ and $y \in T_q(A,\alpha)$, then obviously $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Suppose that $x \in T_\in(A,\alpha)$ and $y \in T_q(A,\alpha)$. Then $A_T(x)+\alpha \geq 2\alpha > 1$, that is, $x \in T_q(A,\alpha)$. It follows that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Similarly, if $x \in T_q(A,\alpha)$ and $y \in T_\in(A,\alpha)$, then $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Now, let $x,y \in F_{\in\vee q}(A,\gamma)$. Then $x \in F_\in(A,\gamma)$ or $x \in F_\in(A,\gamma)$, and $y \in F_\in(A,\gamma)$ or $y \in F_\in(A,\gamma)$. If $x \in F_\in(A,\gamma)$ and $y \in F_\in(A,\gamma)$, then clearly $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. If $x \in F_\in(A,\gamma)$ and $y \in F_\in(A,\gamma)$, then $A_F(x)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A,\gamma)$. It follows that $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Similarly, if $x \in F_q(A,\gamma)$ and $y \in F_\in(A,\gamma)$, then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Finally, assume that $x \in F_\in(A,\gamma)$ and $y \in F_\in(A,\gamma)$. Then $A_F(x)+\gamma \leq 2\gamma < 1$ and $A_F(y)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A,\gamma)$ and $y \in F_q(A,\gamma)$. Therefore $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Consequently, $T_{\in\vee q}(A,\alpha)$ and $F_{\in\vee q}(A,\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$.

**Theorem 3.13** For an intuitionistic fuzzy set $A = (A_T, A_F)$ of $X$, if the nonempty subsets $T_q(A,\alpha)$ and $F_q(A,\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then

$$x \in T_\in(A,\alpha_x), y \in T_\in(A,\alpha_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in T_q(A, \alpha_x \wedge \alpha_y),$$
$$x \in F_\in(A,\gamma_x), y \in F_\in(A,\gamma_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in F_q(A, \gamma_x \vee \gamma_y), \tag{3.6}$$

for all $x,y \in X$, $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$.

Proof. Let $x,y,u,v \in X$, let $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$ and $\gamma_u, \gamma_v \in [\frac{1-k_F}{2}, 1)$ be such that $x \in T_\in(A,\alpha_x)$, $y \in T_\in(A,\alpha_y)$, $u \in F_\in(A,\gamma_u)$ and $v \in F_\in(A,\gamma_v)$. Then $A_T(x) \geq \alpha_x > 1-\alpha_x$, $A_T(y) \geq \alpha_y > 1-\alpha_y$, $A_F(u) \leq \gamma_u < 1-\gamma_u$ and $A_F(v) \leq \gamma_v < 1-\gamma_v$. It follows that $x,y \in T_q(A, \alpha_x \wedge \alpha_y)$ and $u,v \in F_q(A, \gamma_u \vee \gamma_v)$. Since $\alpha_x \wedge \alpha_y \in (0, \frac{1-k_T}{2}]$ and $\gamma_u \vee \gamma_v \in [\frac{1-k_F}{2}, 1)$, we have $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha_x \wedge \alpha_y)$ and $(u|(v|v))|(u|(v|v)) \in F_q(A, \gamma_u \vee \gamma_v)$.

Given a subset $S$ of $X$, consider an intuitionistic fuzzy set $A_S = (A_{S_T}, A_{S_F})$ in $X$ defined by

$$A_S(x) = \begin{cases} (1,0) & \text{if} x \in S \\ (0,1) & otherwise \end{cases}$$

that is,

$$A_{S_T}(x) = \begin{cases} 1 & \text{if} x \in S \\ 0 & otherwise, \end{cases}$$

$$A_{S_F}(x) = \begin{cases} 0 & \text{if} x \in S \\ 1 & otherwise, \end{cases}$$

**Theorem 3.14** A nonempty subset $S$ of $X$ is a subalgebra of $X$ if and only if the intuitionistic fuzzy set $A_S = (A_{S_T}, A_{S_F})$ is an $(\in, \in\vee\, q_{(k_T,k_F)}))$ -intuitionistic fuzzy subalgebra of $X$.

Proof. Let $S$ be a subalgebra of $X$. Then the subsets $T_\in(A_{S_T}, \alpha)$ and $F_\in(A_{S_F}, \gamma)$ are obviously subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Hence $A_S = (A_{S_T}, A_{S_F})$ is an $(\in, \in\vee\, q_{(k_T,k_F)}))$ -intuitionistic fuzzy subalgebra of $X$. Conversely, assume that $A_S = (A_{S_T}, A_{S_F})$ is an $(\in, \in\vee\, q_{(k_T,k_F)}))$ -intuitionistic fuzzy subalgebra of $X$. Let $x,y \in S$. Then

$$A_{S_T}((x|(y|y))|(x|(y|y))) \geq \wedge\ \{A_{S_T}(x), A_{S_T}(y), \frac{1-k_T}{2}\} = 1 \wedge \frac{1-k_T}{2} = \frac{1-k_T}{2},$$

$$A_{S_F}((x|(y|y))|(x|(y|y))) \le \vee \quad \{A_{S_F}(x), A_{S_F}(y), \frac{1-k_F}{2}\}$$
$$= 1 \vee \frac{1-k_F}{2} = \frac{1-k_F}{2},$$

which imply that $A_{S_T}((x|(y|y))|(x|(y|y))) = 1$ and $A_{S_F}((x|(y|y))|(x|(y|y))) = 0$. Hence $(x|(y|y))|(x|(y|y)) \in S$, and so $S$ is a subalgebra of $X$.

**Theorem 3.15** Let $S$ be a subalgebra of $X$. For every $\alpha \in (0, \frac{1-k_T}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, there exists an $(\in, \in\vee\, q_{(k_T,k_F)})$) -intuitionistic fuzzy subalgebra $A = (A_T, A_F)$ of $X$ such that $T_\in(A,\alpha) = S$ and $F_\in(A,\gamma) = S$.

*Proof.* Let $A = (A_T, A_F)$ be an intuitionistic fuzzy set in $X$ defined by

$$A_S(x) = \begin{cases} (\alpha,\gamma) & \text{if} x \in S \\ (0,1) & otherwise \end{cases}$$

that is,

$$A_{S_T}(x) = \begin{cases} \alpha & \text{if} x \in S \\ 0 & otherwise, \end{cases}$$

$$A_{S_F}(x) = \begin{cases} \gamma & \text{if} x \in S \\ 1 & otherwise, \end{cases}$$

Obviously, $T_\in(A,\alpha) = S$ and $F_\in(A,\gamma) = S$. Suppose that $A_T((a|(b|b))|(a|(b|b))) < \wedge \quad \{A_T(a), A_T(b), \frac{1-k_T}{2}\}$ for some $a, b \in X$. Since $Im(A_T) = 2$, it follows that $\{A_T(a), A_T(b), \frac{1-k_T}{2}\} = \alpha$ and $A_T((a|(b|b))|(a|(b|b))) = 0$. Hence $A_T(a) = \alpha = A_T(b)$, and so $a, b \in S$. Since $S$ is a subalgebra of $X$, $(a|(b|b))|(a|(b|b)) \in S$. Thus $A_T((a|(b|b))|(a|(b|b))) = \alpha$, a contradiction. Therefore $A_T((x|(y|y))|(x|(y|y))) \ge \wedge \quad \{A_T(x), A_T(y), \frac{1-k_T}{2}\}$ for all $x, y \in X$. Assume that there exist $a, b \in X$ such that $A_F((a|(b|b))|(a|(b|b))) > \vee \quad \{A_F(a), A_F(b), \frac{1-k_F}{2}\}$. Then $A_F((a|(b|b))|(a|(b|b))) = 1$ and $\vee \quad \{A_F(a), A_F(b), \frac{1-k_F}{2}\} = \gamma$ since $Im(A_F) = 2$. Then $A_F(a) = \gamma = A_F(b)$ and so that $a, b \in S$. Hence $(a|(b|b))|(a|(b|b)) \in S$, and so $A_F((a|(b|b))|(a|(b|b))) = \gamma$, which is a contradiction. Thus $A_F((x|(y|y))|(x|(y|y))) \le \vee \quad \{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for all $x, y \in X$. Therefore $A = (A_T, A_F)$ is an $(\in, \in\vee\, q_{(k_T,k_F)})$) -intuitionistic fuzzy subalgebra of $X$.

## 4. CONCLUSION

In this paper, we explored the interplay between Sheffer stroke operations and intuitionistic fuzzy algebraic structures. Specifically, we introduced and characterized $(\in, \in)$-intuitionistic fuzzy subalgebras and $(q_{(k_T,k_F)}, \in\vee\, q_{(k_T,k_F)})$ -intuitionistic fuzzy subalgebras within the framework of Sheffer stroke Hilbert algebras. By defining and analyzing subsets derived from intuitionistic fuzzy sets, such as $T_\in$, $F_\in$, $T_q$, and $F_q$, we established conditions under which these subsets form subalgebras. We also provided a comprehensive set of theorems and proofs to solidify the theoretical foundation of these structures.

The significance of this work lies in its contribution to the theoretical and practical understanding of non-classical logic systems. The study of intuitionistic fuzzy subalgebras enhances our ability to model uncertainty and vagueness in logical systems, making it particularly relevant for applications in computational models, quantum computing, and fuzzy decision-making frameworks. By leveraging the properties of Sheffer stroke operations, this research offers a unified approach to constructing and analyzing algebraic systems that extend beyond classical Boolean logic.

Future research can build upon the findings of this study in several ways. One potential direction is the exploration of additional types of intuitionistic fuzzy subalgebras and their relationships with other algebraic systems, such as residuated lattices or MV-algebras. Additionally, the integration of intuitionistic fuzzy Sheffer stroke algebras with computational models can provide practical tools for applications in artificial intelligence, particularly in reasoning under uncertainty. Furthermore, extending the results to multi-valued or bipolar fuzzy systems could reveal new insights into the versatility of Sheffer stroke operations in diverse logical and algebraic contexts.

In summary, this work establishes a robust foundation for the study of intuitionistic fuzzy subalgebras in the context of Sheffer stroke Hilbert algebras, paving the way for further theoretical advancements and practical applications in non-classical logic systems.

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