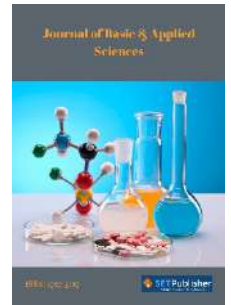




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Walrasian Economic Equilibrium Problems in Convex Regions

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Abstract:

This paper defines Walrasian economic equilibrium in convex regions, introduces the definition of strict-proper quasimonotone functions, and derives some necessary and sufficient conditions for the existence and uniqueness of Walrasian equilibrium vectors in convex regions.

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1. INTRODUCTION

Nash equilibrium [1] is achieved in a game when no player has any incentive for deviating from their own strategy, even if they know the other players' strategies. In economic theory, the Nash equilibrium is used to illustrate that decision-making is a system of strategic interactions based on the actions of other players. It can be used to model economic behaviour to predict the best response to any given situation.

General Equilibrium Theory was developed by the French mathematical economist Leon Walras [2]. Walras argues that, in a perfect competition economy, the participants in the market with a certain number of goods as a supply, will be able to achieve a balance.

He also believes that the total revenue of all production factors and the total revenue from the sale of all consumer goods will be equal under the "perfectly competitive" equilibrium. The essence of this theory is that the capitalist economy can be in a stable equilibrium state. In the capitalist economy, consumers can get the most out of it, the entrepreneur can get the maximum profit, and the owner of the production factors can get the maximum reward.

Walras argues that the process of achieving a balanced price or value is consistent, so price decisions and value decisions are one thing. He believes that various goods, services supplies, demand quantities, and prices are interrelated. A commodity price and quantity changes can cause other goods quantity and price changes. So we can not only study a commodity, a market supply and demand changes, we must also study all the goods, all the market supply and demand changes. Only when all markets are in balance, individual markets can be in a state of equilibrium.

A Walrasian market is an economic model of a market process in which orders are collected into batches of buys and sells and then analyzed to determine a clearing price that will decide the market price. This is also referred to as a call market. Walrasian equilibrium requires that both agents consume their Marshallian demands given prices and also that these demands are compatible. So, what people want to do is set relative prices, find the Marshallian demands of the two agents, and to see whether or not demand equals supply in the markets.

Walrasian general equilibrium price decision thought is illustrated by "mathematical formulas". Through the

solutions of equations, Walras proved that there were a series of market prices and the number of transactions in the market (these prices and quantities are the equilibrium prices and quantities), so that each consumer, entrepreneur and resource owner achieve their purpose, so that society can exist harmoniously and steadily.

Both works of Nash and Walras are Notable Works.

This paper works on Walrasian equilibrium problems as mathematical formulas—variational inequalities over convex regions. Similar approaches can be done for Nash equilibrium problems.

We note that S. C. Fang *et al.* and A. Nagurney discussed the situations of (n-1)- dimensional unit simplex, while I. V. Konnov *et al.* worked on the case of box-constraint set. An (n-1)-dimensional unit simplex and a box-constraint set are both special cases of convex regions.

2. PRELIMINARIES

Let X be a non-empty subset of the n-dimensional Euclidean space R^n and let $E: X \rightarrow D \subseteq R^n$ be a function. A point $x^* \in X$ is said to be a solution of the **variational inequality** $VI(E, X)$ if there holds

$$E(x^*)^T(x - x^*) \geq 0, \forall x \in X.$$

A function $E: X \rightarrow D \subseteq R^n$ is said to be **pseudomonotone** if there holds

$$E(x)^T(y - x) \geq 0 \Rightarrow E(y)^T(y - x) \geq 0, \forall y \in X;$$

strictly pseudomonotone if there holds

$$E(x)^T(y - x) \geq 0 \Rightarrow E(y)^T(y - x) > 0, \forall y \in X, y \neq x.$$

Similar to the proof in [3], one has the following Lemma 2.1.

Lemma 2.1 Let $X \subseteq D \subseteq R^n$ be a convex region and $E: X \rightarrow D \subseteq R^n$ a continuous function, then the variational inequality $VI[E, X]$ has solutions.

Lemma 2.2 Let $X \subseteq R^n$ be a convex region and $E: X \rightarrow D \subseteq R^n$ a continuous and strictly pseudomonotone function, then the variational inequality $VI[E, X]$ has a unique solution.

Proof. Let x^* be a solution of $VI[E, X]$. Then,

$$E(x)^T(x - x^*) \geq 0, \forall x \in X.$$

Due to the strict pseudomonotonicity of $E(x)$ one has

$$E(x)^T(x - x^*) > 0, \forall x \in X, x \neq x^*$$

which means $\forall x^* \in X$ with $x \neq x^*$ is not a solution of $V[E, X]$.

Therefore, $V[E, X]$ has at most one solution. By Lemma 2.1, $V[E, X]$ has a unique solution.

3. AN EXAMPLE OF WALRASIAN ECONOMIC EQUILIBRIUM PROBLEM

This section presents an example of a Walrasian Equilibrium problem in a convex region.

Consider a perfect competition economy with n commodities. Give a price vector $p \in R^n$, the aggregate excess demand function is defined by

$$\begin{aligned} E(p) &= -D(p) + S(p) \\ &= (E_1(p), E_2(p), \dots, E_n(p)), \end{aligned}$$

where D and S are the demand and supply functions, respectively.

Traditionally, a vector p^* is said to be a Walrasian equilibrium price vector [3, 4] if it solves the following variational inequality

$$E(p^*) \geq 0, \forall p \in S.$$

Where

$$S = \{x \in R^n; x_j \geq 0, \sum_{j=1}^n x_j = 1\}$$

is an $(n - 1)$ -dimensional **unit simplex**.

It has been known [5] that a vector p^* is a Walrasian equilibrium price vector if and only if it is a solution of the variational inequality

$$E(p^*)^T(p - p^*) \geq 0, \forall p \in S.$$

If each price of a commodity is involved in the market structure which has a lower positive bound and may have an upper bound, then the prices are assumed to be contained in the **box-constrained set**

$$K = \{x \in R^n; 0 \leq a_j \leq x_j \leq b_j \leq +\infty\},$$

where a_j are constants, b_j are either constants or $+\infty$ ($j = 1, \dots, n$). Then, a vector p^* is a Walrasian equilibrium price vector if it is a solution of the variational inequality over a box-constrained set K [5]:

$$E(p^*)^T(p - p^*) \geq 0, \forall p \in K.$$

Now, the following Walrasian equilibrium problem over a convex feasible region is introduced, which extends the above definition that over a box-constrained set.

Definition 3.1 A price vector p^* is a Walrasian equilibrium price vector if it is a solution to the variational inequality

$$E(p^*)^T(p - p^*) \geq 0, \forall p \in X, \tag{1}$$

where X is a convex region.

An example is given here to illustrate that an economic equilibrium problem may have a (non polytope) convex feasible region.

Example 3.1 The function

$$A = P \frac{p_1(1+p_1)^N}{(1+p_1)^N - 1}$$

is known as the mortgage payment amount that should be paid periodically for N periods on a mortgage amount P at a periodic interest rate of p_1 . The mortgage can be considered as a type of commodity. The price of the mortgage would be the interest rate p_1 , which means p_1 dollars for each one hundred-mortgage for one period. At the time of the mortgage has been paid off, the value p_2 of the house satisfies

$$p_2 \geq B + A = B + P \frac{p_1(1+p_1)^N}{(1+p_1)^N - 1},$$

where B is the down payment in the beginning of the mortgage.

Consider the function

$$p_2 = f(p_1) = B + P \frac{p_1(1+p_1)^N}{(1+p_1)^N - 1}.$$

For a perfect competition economy with n commodities p_1, p_2, \dots, p_n , let

$$\begin{aligned} X &= \{p \in R^n : 0 \leq p_1 \leq g(p_2), \\ &0 \leq a_j \leq p_j \leq b_j \leq +\infty, j = 2, 3, \dots, n\} \end{aligned}$$

where $p = (p_1, p_2, \dots, p_n) \in R^n$ is the price vector. If $b_j < +\infty$ ($j = 2, 3, \dots, n$), then X is a non-polytope convex region.

4. SOME NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF WALRASIAN EQUILIBRIUM PRICE VECTORS

A price vector $p^* \in X$ is said to be a **stationary price vector** of the Walrasian equilibrium problem (1) if there holds

$$E(p)^T(p - p^*) \geq 0, \forall p \in X.$$

A. Daniilidis and N. Hadjisavvas [6] introduced the following definition of the proper quasimonotonicity.

The function $E: X \rightarrow D \subseteq R^n$ is said to be **properly quasimonotone** if $\forall p_i \in X, j \in \{1, 2, \dots, m\}, \forall p \in \text{con}(p_1, p_2, \dots, p_m), \exists j \in \{1, 2, \dots, m\}$ such that

$$E(p_j)^T(p_j - p) \geq 0.$$

The following Theorem 4.1 gives a necessary and sufficient condition for the existence of the Walrasian equilibrium price vector.

Theorem 4.1 There exists a stationary price vector p^* for the Walrasian equilibrium problem (1) if and only if $E: X \rightarrow D \subseteq R^n$ is properly quasimonotone.

Proof. The proof of the sufficient is from A. Daniilidis and N. Hadjisavvas [7], and the necessity can be found in R. John [8].

We introduce the definition of **strict-proper quasimonotonicity**.

Definition 4.1 The function $E: X \rightarrow D \subseteq R^n$ is said to be **strict-properly quasimonotone** if $\forall p_i \in X, i \in \{1, 2, \dots, m\}, \forall p \in \text{con}(p_1, p_2, \dots, p_m)$ with $p \neq p_i, i \in \{1, 2, \dots, m\}, \exists j \in \{1, 2, \dots, m\}$ such that

$$E(p_j)^T(p_j - p) > 0.$$

Theorem 4.2 If $E: X \rightarrow D \subseteq R^n$ is strict-properly quasimonotone, then E is strictly pseudomonotone, i.e., $\forall p, q \in X, p \neq q,$

$$E(p)^T(q - p) \geq 0 \Rightarrow E(q)^T(q - p) > 0.$$

Proof. Let

$$E(p)^T(q - p) \geq 0, p \neq q.$$

Taking

$$r = \frac{1}{2}(p + q),$$

due to the strict-proper quasimonotonicity of $E,$

$$E(q)^T(p - q) < 0.$$

In the following Definition 4.2 of “generalized B -preaffinelike function”, we assume that B is a given pointed convex cone. Ref. [9] proves that the definition of generalized B -preaffineness is non-trivial.

Definition 4.2 A function $E: X \rightarrow D \subseteq R^n$ is said to be generalize B -preaffinelike on X if $\forall x_1, x_2 \in D, \forall \alpha \in R, \exists u \in B, \exists x_3 \in X, \exists \tau \in R \setminus \{0\}$ such that

$$u + \alpha E(x_1) + (1 - \alpha)E(x_2) = \tau E(x_3).$$

The following Theorem 4.3 gives necessary and sufficient conditions for the existence and uniqueness of the Walrasian equilibrium price vector and of the stationary price vector.

Theorem 4.3 Let $E: X \rightarrow D \subseteq R^n$ be continuous and generalize B -preaffinelike on $X,$ where B is the first quadrant. Then the following are equivalent:

- (a) $E: X \rightarrow D \subseteq R^n$ is strict-properly quasimonotone;
- (b) There exists a unique stationary price vector p^* for the Walrasian equilibrium problem (1);
- (c) There exists a unique Walrasian equilibrium price vector for the Walrasian equilibrium problem (1).

Proof. (a) \Rightarrow (b). If $E: X \rightarrow D \subseteq R^n$ is strict-properly quasimonotone, then E is properly quasimonotone. By Theorem 4.1 and 4.2, the Walrasian equilibrium problem (1) has stationary price vectors.

Assume p^* and p^{**} are both stationary price vectors i.e., $\forall p \in X,$

$$E(p)^T(p - p^*) = 0,$$

$$E(p)^T(p - p^{**}) = 0.$$

So, $\forall \lambda \in [0, 1],$

$$E(p)^T(p - (\lambda p^* + (1 - \lambda)p^{**})) = 0, \forall p \in X.$$

Due to the strict-proper quasimonotonicity, one must have

$$p^* = p^{**}.$$

(b) \Rightarrow (c). From Theorem 4.2.

(c) \Rightarrow (a). Assume that there exists a unique Walrasian equilibrium price vector for the problem (1), then by Theorem 4.1 and 4.2, $E: X \rightarrow D \subseteq R^n$ is properly quasimonotone.

If $E: X \rightarrow D \subseteq R^n$ is not strict-properly quasimonotone, then $\exists p_1, p_2, \dots, p_m \in X$, and $\exists p_0 \in \text{con}(p_1, p_2, \dots, p_m)$, such that

$$E(p_j)^T(p_j - p_0) = 0, \forall j \in \{1, 2, \dots, n\}.$$

Take

$$j(0) \in \{1, 2, \dots, m\} \text{ with } p_{j(0)} \neq p_0,$$

then

$$E(p_{j(0)})^T((\lambda p_0 + (1-\lambda)p_{j(0)}) - p_{j(0)}) = 0. \quad (2)$$

So, $p_{j(0)}$ is a solution the variational inequality

$$E(p^*)^T(p - p^*) \geq 0, p \in [p_0, p_{j(0)}]. \quad (3)$$

where $[p_0, p_{j(0)}]$ is the line segment

$$[p_0, p_{j(0)}] = \{p \in X : p = \lambda p_0 + (1-\lambda)p_{j(0)}, \lambda \in [0, 1]\}.$$

Combining (2) and Theorem 4.2,

$$E(p)^T(p - p_{j(0)}) > 0, p \in [p_0, p_{j(0)}], p \neq p_{j(0)}.$$

Therefore, $\forall p_{j(1)} \in [p_0, p_{j(0)}], p_{j(1)} \neq p_{j(0)}$, one has

$$E(p_{j(1)})^T(p_{j(0)} - p_{j(1)}) < 0,$$

which means $\forall p_{j(1)} \in [p_0, p_{j(0)}]$ with $p_{j(1)} \neq p_{j(0)}$ is not a solution of the variational inequality (3).

Hence, $p_{j(0)}$ is the unique solution to the variational inequality (3).

So, if given $p \in [p_0, p_{j(0)}]$, and $p \neq p_{j(0)}$, then

$$E(p)^T((\lambda p_0 + (1-\lambda)p_{j(0)}) - p) < 0, 0 \leq \lambda \leq 1.$$

i.e.,

$$E(p)^T(p - (\lambda p_0 + (1-\lambda)p_{j(0)})) > 0.$$

If $p = p_{j(0)}$, from (2)

$$E(p)^T(p - (\lambda p_0 + (1-\lambda)p_{j(0)})) = 0.$$

Therefore

$$E(p)^T(p - (\lambda p_0 + (1-\lambda)p_{j(0)})) \geq 0, \forall p \in [p_0, p_{j(0)}].$$

Then, for given $\lambda \in (0, 1]$,

$$\lambda p_0 + (1-\lambda)p_{j(0)}$$

is a solution of the dual variational inequality

$$E(p)^T(p - p^*) \geq 0, p \in [p_0, p_{j(0)}].$$

Therefore $\lambda p_0 + (1-\lambda)p_{j(0)}$ is a solution of the variational inequality (3). This contradicts to the fact that $p_{j(0)}$ is the unique solution to the variational inequality (3).

Consequently, $E: X \rightarrow D \subseteq R^n$ is strict-properly quasimonotone.

5. CONCLUSION

Ref. [3, 4] worked on Walrasian equilibrium over an (n-1) dimensional unit simplex, while Ref. [5] dealt with Walrasian equilibrium over a box-constraint set. This paper obtains some necessary and sufficient conditions for the existence and uniqueness of the solution of a Walrasian equilibrium problem over a convex region, which means that, in a pure exchange economy (perfect competition economy), the participants in the market with a certain number of goods as a supply, will be able to achieve a balance.

More discussions may be carried out for solutions of the corresponding variational inequalities. We remark that the analytic center cutting plane methods proposed in Ref. [20] can be used to find constructive solutions of our variational inequalities.

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