Published by SET Publisher

Journal of Basic & Applied Sciences

ISSN (online): 1927-5129

On New Simultaneous Generalizations of Well-Known Fixed Point Theorems

Wei-Shih Du*

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 824004, Taiwan

Article Info:

Keywords:
MT-function (R-function),
Nadler's fixed point theorem,
Banach contraction principle,
Simultaneous generalization,
Kannan's fixed point theorem,
Chatterjea's fixed point theorem,
Berinde-Berinde's fixed point theorem,
Mizoguchi-Takahashi's fixed point theorem.

Timeline:
Received: December 22, 2024
Accepted: January 16, 2025
Published: February 12, 2025

Citation: Du W-S. On New Simultaneous Generalizations of Well-Known Fixed Point Theorems. J Basic Appl Sci 2025; 21: 42-46.

DOI: https://doi.org/10.29169/1927-5129.2025.21.05

*Corresponding Author
E-mail: wsdu@mail.nknu.edu.tw

Abstract:

The primary goal of this work is to establish some new fixed point theorems and new simultaneous generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

2020 Mathematics Subject Classification: 47H09, 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Let (X,d) be a metric space. For $x \in X$ and a subset A of X, define

$$d(x,A) = \inf_{y \in A} d(x,y).$$

Denote by $N(X)$ the family of all nonempty subsets of X and $CB(X)$ the class of all nonempty closed and bounded subsets of X. A function $H: CB(X) \times CB(X) \to 0,\infty)$ defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\}$$

is said to be the Hausdorff metric on $CB(X)$ induced by the metric d on X. A point v in X is said to be a *fixed point* of of a mapping T if $v \in Tv$ (when $T: X \to N(X)$ is a multivalued mapping) or $Tv = v$ (when $T: X \to X$ is a single-valued mapping). The set of fixed points of T is denoted by $F(T)$. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

Recall that a function ϕ: $[0,+\infty) \to [0,1)$ is said to be an MT *-function* (or R *-function*) [4-7] if $1 > \limsup_{s \to t^+} \varphi(s) := \inf_{\varepsilon > 0} \sup_{t<s<t+\varepsilon} \varphi(s)$ for all $t \in [0,+\infty)$. Clearly, if ϕ: $[0,+\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then ϕ is a MT -function. So the set of MT -functions is a rich class.

In 2012, Du [5] established the following powerful characterizations of MT -functions.

Theorem 1.1 (see [5, Theorem 2.1]). Let ϕ: $[0,+\infty) \to [0,1)$ be a function. Then the following statements are equivalent.

(a). ϕ is an MT -function.

(b). For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\phi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.

(c). For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\phi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.

(d). For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\phi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.

(e). For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\phi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.

(f). For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \phi(x_n) < 1$.

(g). ϕ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \phi(x_n) < 1$.

In 2007, M. Berinde and V. Berinde [2] established the following interesting fixed point theorem.

Theorem 1.2 (M. Berinde and V. Berinde [2]) Let (X,d) be a complete metric space, $T: X \to \mathrm{CB}(X)$ be a multivalued mapping, ϕ: $[0,+\infty) \to [0,1)$ be an MT -function and $L \ge 0$. Assume that

$\mathcal{H}(Tx,Ty) \le \phi(d(x,y))d(x,y) + Ld(y,Tx)$ for all $x,y \in X$.

Then $F(T) \neq \emptyset$.

If $L = 0$ in Theorem 1.2, then we can obtain Mizoguchi-Takahashi's fixed point theorem [9].

Theorem 1.3 (Mizoguchi and Takahashi [9]) Let (X,d) be a complete metric space, ϕ: $[0,+\infty) \to [0,1)$ be an MT -function and $T: X \to CB(X)$ be a multivalued mapping. Assume that

$\mathcal{H}(Tx,Ty) \le \phi(d(x,y))d(x,y)$ for all $x, y \in X$.

Then $F(T) \neq \emptyset$.

If $\phi(t) = \lambda \in [0,1)$ for all $t \in [0,+\infty)$ in Theorem 1.3, then we can obtain Nadler's fixed point theorem [10] which extends Banach contraction principle [1] from single-valued mappings to multivalued mappings.

Theorem 1.4 (Nadler [10]) Let (X,d) be a complete metric space and $T: X \to CB(X)$ be a λ -contraction; that is, there exists a nonnegative number $\lambda < 1$ such that

$$\mathrm{H}(Tx,Ty) \le \lambda d(x,y) \text{ for all } x,y \in X.$$

Then $F(T) \neq \emptyset$.

In 1969, Kannan [8] established the following fixed point theorem for single-valued mappings:

Theorem 1.5 (Kannan [8]) Let (X,d) be a complete metric space and $T: X \to X$ be a self-mapping on X. Suppose that there exists $\gamma \in \left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le \gamma\big(d(x,Tx) + d(y,Ty)\big) \quad \text{for all } x,y \in X.$$

Then T admits a unique fixed point in X.

In 1972, Chatterjea proved so-called the Chatterjea's fixed point theorem [3] as follows:

Theorem 1.6 (Chatterjea [3]) Let (X,d) be a complete metric space and $T: X \to X$ be a self-mapping on X. Suppose that there exists $\gamma \in \left[0,\frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le \gamma(d(x,Ty)+d(y,Tx)) \quad \text{for all } x,y \in X.$$

Then T admits a unique fixed point in X.

In this work, we will establish some new fixed point theorems and new simultaneous generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

2. MAIN RESULTS

Theorem 2.1 Let (X,d) be a complete metric space, $T: X \to \mathrm{CB}(X)$ be a multivalued mapping and g: $X \to [0,+\infty)$ be a function. Suppose that

(D). there exists an MT -function μ : $[0,+\infty) \to [0,1)$ such that

$$\mathcal{H}(Tx,Ty) \le \mu(d(x,y))\, max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\} + g(y)d(y,Tx)$$

for all $x,y \in X$.

Then $F(T) \neq \emptyset$.

Proof. Define $\kappa: [0,+\infty) \to (0,1)$ by

$$\kappa(t) = \frac{\mu(t)+1}{2} \quad \text{for all } t \in [0,+\infty).$$

Then we have

$$0 \le \mu(t) < \kappa(t) < 1 \quad \text{for all } t \in [0,+\infty).$$

First, we note that condition (D) implies that for each $x \in X$ with $x \notin Tx$, it holds

$$d(y,Ty) < \kappa(d(x,y))\, max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)}{2}\right\} \quad \text{for all } y \in Tx. \quad (2.1)$$

Let $x_1 \in X$ with $x_1 \notin Tx_1$ and $x_2 \in Tx_1$. Thus $x_1 \neq x_2$ or $d(x_1,x_2) > 0$. By (2.1), we have

$$d(x_2,Tx_2)$$

$$< \kappa(d(x_1,x_2))\, max\left\{d(x_1,x_2), \frac{d(x_1,Tx_1)+d(x_2,Tx_2)}{2}, \frac{d(x_1,Tx_2)}{2}\right\}$$

$$\le \kappa(d(x_1,x_2))\, max\left\{d(x_1,x_2), \frac{d(x_1,x_2)+d(x_2,Tx_2)}{2}, \frac{d(x_1,Tx_2)}{2}\right\}. \quad (2.2)$$

If $x_2 \in Tx_2$, then $x_2 \in F(T)$ and we are done. Suppose $x_2 \notin Tx_2$. By (2.2), there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) < \kappa(d(x_1,x_2))\, max\left\{d(x_1,x_2), \frac{d(x_1,x_2)+d(x_2,x_3)}{2}, \frac{d(x_1,x_3)}{2}\right\}. \quad (2.3)$$

Assume $d(x_1,x_2) < d(x_2,x_3)$. By (2.3), we get

$$d(x_2,x_3) < \kappa(d(x_1,x_2))d(x_2,x_3) < d(x_2,x_3),$$

which leads a contradiction. Hence it must be $d(x_2,x_3) \le d(x_1,x_2)$ and (2.3) deduces

$$d(x_2,x_3) < \kappa(d(x_1,x_2))d(x_1,x_2).$$

Since $x_2 \neq x_3$, $d(x_2,x_3) > 0$. Using condition (D), we obtain

$$d(x_3,Tx_3) < \kappa\big(d(x_2,x_3)\big)$$

$$\kappa(d(x_2,x_3))\, max\left\{d(x_2,x_3), \frac{d(x_2,x_3)+d(x_3,Tx_3)}{2}, \frac{d(x_2,Tx_3)}{2}\right\}.$$

If $x_3 \in Tx_3$, then $x_2 \in F(T)$ and we finish this proof. Otherwise, there exists $x_4 \in Tx_3$ such that

$$d(x_3,x_4) < \kappa\big(d(x_2,x_3)\big)\, max\left\{d(x_2,x_3), \frac{d(x_2,x_3)+d(x_3,x_4)}{2}, \frac{d(x_2,x_4)}{2}\right\}. \quad (2.4)$$

If $d(x_2,x_3) < d(x_3,x_4)$, then, by (2.4), we have

$$d(x_3,x_4) < \kappa(d(x_2,x_3))d(x_3,x_4) < d(x_3,x_4),$$

a contradiction. So it must be $d(x_3,x_4) \le d(x_2,x_3)$ and hence

$$d(x_3,x_4) < \kappa(d(x_2,x_3))d(x_2,x_3).$$

By induction, we can obtain a sequence $\{x_n\}$ in X satisfying for each $n \in N$,

(i). $d(x_n,x_{n+1}) > 0$,

(ii). $x_{n+1} \in Tx_n$,

(iii). $d(x_{n+1},x_{n+2}) < \kappa(d(x_n,x_{n+1}))d(x_n,x_{n+1})$.

Since $\kappa(t) < 1$ for all $t \in (0,\infty)$, by (iii), the sequence $\{d(x_n,x_{n+1})\}$ is strictly decreasing in $(0,\infty)$. Then

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n,x_{n+1}) \ge 0 \; exists$$

Since μ is an MT -function, by applying Theorem 1.1,

$$0 \le \sup_{n\in\mathbb{N}} \mu(d(x_n,x_{n+1})) < 1,$$

which yields

$$0 < \sup_{n\in\mathbb{N}} \kappa(d(x_n,x_{n+1})) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \mu(d(x_n,x_{n+1}))\right] < 1,$$

Let $\beta := \sup_{n\in\mathbb{N}} \kappa(d(x_n,x_{n+1}))$. So $\beta \in (0,1)$. Using (iii) again, we obtain

$$d(x_{n+1},x_{n+2}) < \kappa(d(x_n,x_{n+1}))d(x_n,x_{n+1})$$

$$\leq \beta d(x_n, x_{n+1})$$
$$\leq \cdots$$
$\leq \beta^n d(x_1, x_2)$ for each $n \in \mathbb{N}$.

We now claim that $\{x_n\}$ is a Cauchy sequence in X. Let $w_n = \frac{\beta^{n-1}}{1-\beta} d(x_1, x_2)$, $n \in \mathbb{N}$. For m, $n \in N$ with $m > n$, we obtain

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < w_n. \quad (2.5)$$

Since $\beta \in (0,1)$, $\lim_{n\to\infty} w_n = 0$. From (2.5), we get

$$\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0,$$

which show that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. For any $n \in \mathbb{N}$, from condition (D) again, we have

$$d(x_{n+1}, Tv) \leq \mathrm{H}(Tx_n, Tv)$$
$$\leq \mu(d(x_n, v))\, max \left\{ \begin{matrix} d(x_n, v), \dfrac{d(x_n, x_{n+1}) + d(v, Tv)}{2}, \\ \dfrac{d(x_n, Tv) + d(v, x_{n+1})}{2} \end{matrix} \right\}$$
$$+ g(v) d(v, x_{n+1}).$$

Since the function $x \mapsto d(x, Tv)$ is continuous, by taking the limit as $n \to \infty$ on both sides of the last inequality, we obtain

$$d(v, Tv) \leq \frac{d(v, Tv)}{2},$$

which implies $d(v, Tv) = 0$. By the closedness of Tv, we obtain $v \in F(T)$. The proof is completed.

Remark 2.2 Theorem 2.1 simultaneously generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

The following result follows directly from Theorem 2.1.

Corollary 2.3 Let (X, d) be a complete metric space, $T: X \to \mathrm{CB}(X)$ be a multivalued mapping, $g: X \to [0, +\infty)$ be a function and $\mu : [0, +\infty) \to [0,1)$ be an MT -function. Suppose that T satisfies one of the following conditions:

(1).$\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2} \right\} + g(y) d(y, Tx)$ for all $x, y \in X$,

(2).$\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ d(x, y), \frac{d(x,Ty)+d(y,Tx)}{2} \right\} + g(y) d(y, Tx)$ for all $x, y \in X$,

(3i).$\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2} \right\} + g(y) d(y, Tx)$ for all $x, y \in X$,

(4). $\mathcal{H}(Tx, Ty) \leq \mu(d(x, y)) d(x, y) + g(y) d(y, Tx)$ for all $x, y \in X$,

(5). $\mathcal{H}(Tx, Ty) \leq \frac{1}{2}\mu(d(x, y))(d(x, Tx) + d(y, Ty)) + g(y) d(y, Tx)$ for all $x, y \in X$,

(6). $\mathcal{H}(Tx, Ty) \leq \frac{1}{2}\mu(d(x, y))(d(x, Ty) + d(y, Tx)) + g(y) d(y, Tx)$ for all $x, y \in X$.

Then $F(T) \neq \emptyset$.

If $g(z) = 0$ for all $z \in X$ in Theorem 2.1, then we obtain the following conclusions immediately.

Corollary 2.4 Let (X, d) be a complete metric space, $T: X \to \mathrm{CB}(X)$ be a multivalued mapping. Suppose that there exists an MT -function $\mu : [0, +\infty) \to [0,1)$ such that

$$\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ \begin{matrix} d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \\ \frac{d(x,Ty)+d(y,Tx)}{2} \end{matrix} \right\}$$

for all $x, y \in X$. Then $F(T) \neq \emptyset$.

Remark 2.5 Corollary 2.4 simultaneously generalize Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem.

Corollary 2.6 Let (X, d) be a complete metric space, $T: X \to \mathrm{CB}(X)$ be a multivalued mapping and $\mu : [0, +\infty) \to [0,1)$ be an MT -function. Suppose that T satisfies one of the following conditions:

(1). $\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2} \right\}$ for all $x, y \in X$,

(2). $\mathcal{H}(Tx, Ty) \leq \mu(d(x, y))\, max \left\{ d(x, y), \frac{d(x,Ty)+d(y,Tx)}{2} \right\}$ for all $x, y \in X$,

(3).$\mathcal{H}Tx, Ty) \leq \mu(d(x, y))\, max \left\{ \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2} \right\}$ for all $x, y \in X$,

(4). $\mathcal{H}(Tx, Ty) \leq \mu(d(x, y)) d(x, y)$ for all $x, y \in X$,

(5). $\mathcal{H}(Tx, Ty) \leq \frac{1}{2}\mu(d(x, y))(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$, (6). $\mathcal{H}(Tx,Ty) \le \frac{1}{2}\mu(d(x,y))(d(x,Ty)+d(y,Tx))$ for all $x,y \in X$.

Then $F(T) \neq \emptyset$.

As an application of Theorem 2.1 (or Corollary 2.4), we establish the following existence and uniqueness of fixed point theorem for single-valued mappings.

Corollary 2.7 Let (X,d) be a complete metric space, $T: X \to X$ be a sefmapping and $\mu : [0,+\infty) \to [0,1)$ be an MT -function. Suppose that T satisfies one of the following conditions:

(1). $d(Tx,Ty) \le \mu(d(x,y))\, max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ for all $x,y \in X$,

(2). $d(Tx,Ty) \le \mu(d(x,y))\, max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}\right\}$ for all $x,y \in X$,

(3). $d(Tx,Ty) \le \mu(d(x,y))\, max\left\{d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ for all $x,y \in X$,

(4). $d(Tx,Ty) \le \mu(d(x,y))\, max\left\{\frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ for all $x,y \in X$,

(5). $d(Tx,Ty) \le \mu(d(x,y))d(x,y)$ for all $x,y \in X$,

(6). $d(Tx,Ty) \le \frac{1}{2}\mu(d(x,y))(d(x,Tx)+d(y,Ty))$ for all $x,y \in X$,

(7). $d(Tx,Ty) \le \frac{1}{2}\mu(d(x,y))(d(x,Ty)+d(y,Tx))$ for all $x,y \in X$.

Then T admits a unique fixed point in X.

Proof. Clearly, each of conditions (2)-(7) implies condition (1), so we only have to prove that T admits a unique fixed point in X if condition (1) holds. Assume that condition (1) holds. Applying Theorem 2.1 (or Corollary 2.4), $F(T) \neq \emptyset$. We now want to show that $F(T)$ is a singleton set. Assume there exist $u,v \in F(T)$ with $u \neq v$. So $d(u,v) > 0$. By (1), we get

$$d(u,v) = d(Tu,Tv) \le \mu(d(u,v))d(u,v) < d(u,v)$$

a contradiction. Therefore $F(T)$ is a singleton set and T has a unique fixed point in X. The proof is completed.

Remark 2.8 Corollary 2.7 simultaneously improves Banach contraction principle, Kannan's fixed point theorem and Chatterjea's fixed point theorem for single-valued mappings.

ACKNOWLEDGEMENT

The author is partially supported by Grant No. NSTC 113-2115-M-017-004 of the National Science and Technology Council of the Republic of China.

REFERENCES

[1] Banach S. Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. Fund Math 1922; 3: 133-181. https://doi.org/10.4064/fm-3-1-133-181

[2] Berinde M, Berinde V. On a general class of multi-valued weakly Picard mappings. J Math Anal Appl 2007; 326: 772-782. https://doi.org/10.1016/j.jmaa.2006.03.016

[3] Chatterjea SK. Fixed-point theorems. C.R. Acad. Bulgare Sci 1972; 25: 727-730.

[4] Du W-S. Some new results and generalizations in metric fixed point theory. Nonlinear Anal 2010; 73: 1439-1446. https://doi.org/10.1016/j.na.2010.05.007

[5] Du W-S. On coincidence point and fixed point theorems for nonlinear multivalued maps. Topology and its Applications 2012; 159: 49-56. https://doi.org/10.1016/j.topol.2011.07.021

[6] Du W-S. New existence results of best proximity points and fixed points for $\mathrm{MT}(\lambda)$ -functions with applications to differential equations. Linear and Nonlinear Analysis 2016; 2(2): 199-213.

[7] Du W-S. Simultaneous generalizations of fixed point theorems of Mizoguchi-Takahashi type, Nadler type Banach type, Kannan type and Chatterjea type. Nonlinear Analysis and Differential Equations 2017; 5(4): 171-180. https://doi.org/10.12988/nade.2017.723

[8] Kannan R. Some results on fixed point- II. Amer Math Monthly 1969; 76: 405-408. https://doi.org/10.1080/00029890.1969.12000228

[9] Mizoguchi N, Takahashi W. Fixed point theorems for multivalued mappings on complete metric spaces. J Math Anal Appl 1989; 141: 177-188. https://doi.org/10.1016/0022-247X(89)90214-X

[10] Nadler SB. Jr. Multi-valued contraction mappings. Pacific J Math 1969; 30: 475-488. https://doi.org/10.2140/pjm.1969.30.475