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A Study on Fractional Integral Inequalities for Trigonometric and Exponential Trigonometric-convex Functions

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Abstract:

Inequalities involving fractional operators have also been an active area of research. These inequalities play a crucial role in establishing bounds, estimates, and stability conditions for solutions to fractional integrals. In this paper, firstly we establish these new identities for the case of twice differentiable functions and Caputo-Fabrizio fractional integrals. By utilizing these new identities, novel inequalities are obtained for trigonometric convex functions, and exponential trigonometric convex functions and exponential trigonometric convex functions. It is expected that the outcomes of this research will point to new developments in the study of fractional calculus.

1991 Mathematics Subject Classification: 26D07; 26D10; 26D15.

1. INTRODUCTION

The theory of inequalities and fractional calculus have indeed been subjects of extensive research and investigation in mathematics. Fractional calculus enriches our mathematical toolkit by extending classical ideas to non-integer orders, leading to a deeper understanding of various phenomena and providing more accurate models for real-world processes. The use of fractional operators in mathematical modeling offers a more nuanced and realistic approach to understanding systems with memory and non-local interactions. Researchers and practitioners continue to explore and refine these models to address specific challenges in diverse fields. As examples, consider the third-order BVP with multistrip multi-point conditions [1], the fractional HIV model with the Mittag-Leffler-type kernel [2], the fractional dynamics of the mumps virus [3], the fractal-fractional version of the AH1N1/09 virus along with the fractional Caputo-type version [4].

There are many other important inequalities in mathematics the Hermite-Hadamard inequality is one of these. The Hermite-Hadamard inequality is a mathematical inequality named after Charles Hermite and Jacques Henri Hadamard. We mentioned trapezoidal inequalities arising from the Hermite-Hadamard inequality. These could be specific applications or refinements of the Hermite-Hadamard inequality involving trapezoidal functions or techniques. In the few decades research in this area likely involves exploring different aspects of the Hermite-Hadamard inequality, such as generalizations, applications to various functions, and connections with other mathematical concepts. The introduction of these fractional integrals into inequalities, such as the Hermite-Hadamard inequality and trapezoidal inequalities, indicates a broader trend in incorporating fractional calculus. Additionally, Yang *et al.* [5] established the midpoint-type inequalities for the case of twice differentiable functions and given application to matrix inequalities. Junjua *et al.* [6] obtained the trapezoidal-type inequalities for the case of twcie differentiable functions using the C-F (Caputao-Fabrizio) fractional integrals. Hasan *et al.* [7] proved the trapezoidal-type inequalities for the case of twice differentiable functions through the Conformable fractional integrals. Dumitru *et al.* [8] derived the trapezoidal-type inequalities for the case of differentiable functions employing the generalized fractional integrals. Fatih *et al.* [9] presented the new error bounds on trapezoidal-type inequalities utilizing the Conformable fractional integrals. Moreover, Sarikaya *et al.* and Mohammed *et al.* obtained the trapezoidal-type inequalities using fractional operator in [10, 11]. Furthermore, Waqar *et al.* [12] generalized the trapezoidal-type inequalities for h-Godunova-Levin. For more, information on trapezoidal-type inequalities for fractional operators see these articles [13, 14].

The concept of fractional derivatives has gained significant attention and interest from researchers in recent years. The development of fractional integration operators and their applications highlight the interdisciplinary nature of this mathematical concept. Khalil *et al.* [15] established a novel well-behaved fractional derivative, called the conformable derivative, using simply the concept of the fundamental limit formulation for derivatives. The derivative Caputo yields some applicable features that are not obtained for the Riemann-Liouville operators. However, Jamshed *et al.* [16] obtained the new error bounds on Ostrowski-type inequalities for preinvex function. Budak *et al.* [17] proved the Ostrowski-type inequalities using generalized fractional integrals. Budak *et al.* [18] generalized Ostrowski-type inequalities employing the co-ordinated convex functions for fractional integrals. Qayyum *et al.* [19] derived the Ostrowski-type inequalities for weight functions. Saleem et .al [20] presented the Ostrowski-type inequalities for quadratic kernel. For more, information on these type of inequalities for fractional operator see these articles [21-24].

On the other hand the concept of convexity has a rich history in ancient times. The theory of convexity has found extensive use and significance in a wide range of modern mathematical disciplines, including real analysis, functional analysis, linear algebra, and optimization. Theory of inequalities, where the idea of convexity is crucial for enhancing the estimation bounds of various kinds of integral inequalities [25, 26].

2. PRELIMINARIES

Definition 1 If $f: I \to R$ is said to be convex on I if:

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b).$$

for all $(a, b) \in I$ and $t \in [0,1]$.

Definition 2 [27] If $f: I \to R$ is said to be h -convex function or $\left(f \in SX(h, I)\right)$ if the inequality holds:

$$f(ta + (1-t)b) \le h(\alpha)f(a) + h(1-\alpha)f(b),$$

for all $a, b \in I$, $\alpha \in (0,1)$ and $t \in [0,1]$.

Definition 3 [28] If $f: I \to R$ is said to be Trigonometrically-convex function if the inequality holds:

$$f(ta + (1-t)b) \leq sin\left(\frac{\pi t}{2}\right) f(a) + cos\left(\frac{\pi t}{2}\right) f(b),$$

for all $a, b \in I$ and $t \in [0,1]$. Additional every Trigonometrically-convex function is h -convex function with $h(t) = sin\left(\frac{\pi t}{2}\right)$.

Definition 4 [29] If $f: I \subseteq R \to R$ is said to be Exponential trigonometric-convex function if the inequality holds:

$$f(ta + (1-t)b) \leq \frac{sin\left(\frac{\pi t}{2}\right)}{e^{1-t}} f(a) + \frac{cos\left(\frac{\pi t}{2}\right)}{e^{t}} f(b),$$

for all $a, b \in I$ and $t \in [0,1]$. Additional every Exponential trigonometric convex function is trigonometrically convex function.

Both the Caputo and Riemann-Liouville fractional derivatives have found applications in various scientific and engineering disciplines, providing a powerful tool for modeling and analyzing complex systems with memory effects or nonlocal interactions.

Definition 5 [30] Suppose $f \in L[a,b]$. The left$whitei$and right-sided$whitei$Riemann-Liouville fractional$whitei$integrals of order $\alpha > 0$ defined by:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

where $\Gamma(\alpha)$ is the gamma$whitei$function$whitei$and $I_{a^+}^0 f(t) = I_{b^-}^0 f(t) = f(t)$.

Definition 6 [31] Let $H^1(a,b)$ be the$whitei$Sobolev$whitei$space of order$whitei$one defined as

$$H^1(a,b) = \left\{g \in L^2(a,b) : g' \in L^2(a,b)\right\}.$$

where

$$L^2(a,b) = \left\{g(z) : \left(\int_a^b g^2(z) dz\right)^{\frac{1}{2}} < \infty\right\}.$$

Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then$whitei$the notion$whitei$of left derivative in the sense$whitei$of Caputo-Fabrizio is$whitei$defined as:

$$\left({}_{a}^{CFD}D^{\alpha} f\right)(x) = \frac{\beta(\alpha)}{1-\alpha} \int_a^x f'(t) e^{\frac{-\alpha(x-t)^{\alpha}}{1-\alpha}} dt.$$

$x > \alpha$ and the associated$whitei$integral operator is

$$\left({}_{a}^{CF}I^{\alpha} f\right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_a^x f(t) dt.$$

where $\beta(\alpha) > 0$ is the$whitei$normalization$whitei$function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0$ and $\alpha = 1$, the left$whitei$derivative is$whitei$defined as$whitei$follows, respectively

$$\left({}_{a}^{CFD}D^{0} f\right)(x) = f'(x),$$

$$\left({}_{a}^{CFD}D^{1} f\right)(x) = f(x) - f(a).$$

For the right derivative operator

$$\left({}_{b}^{CFD}D^{\alpha} f\right)(x) = \frac{\beta(\alpha)}{1-\alpha} \int_x^b f'(t) e^{\frac{-\alpha(t-x)^{\alpha}}{1-\alpha}} dt.$$

$x < b$ and the$whitei$associated integral$whitei$operator is

$$\left(CFI_b^{\alpha} f\right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)} \int_x^b f(t) dt.$$

where $\beta(\alpha) > 0$ is a normalization function that satisfies $\beta(0) = \beta(1) = 1$

Motivated by the ongoing research, the main goal.in this.paper is to establish a new integral identities using the fractional operator. By using these new identities to proved the error bounds of Ostrowski-type, and trapezoidal-type inequalities for C-F(Caputao-Fabrizio) fractional operator. Based on these identities, we developed the novel inequalities are obtained for trigonometric convex functions and exponential trigonometric convex functions.

3. OSTROWSKI-TYPE INEQUALITIES FOR TRIGONOMETRIC-CONVEX FUNCTION

In this section, we prove the Ostrowski-type inequalities for trigonometric convex functions. We apply the Caputo-Fabrizio operator to attain these inequalities. To obtain Caputo-Fabrizio fractional integral Ostrowski-type inequalities, we require the following Lemma:

Lemma 1 Suppose $I \subseteq R$, $f: I \to R$ be a twice differentiable mapping on I^o where $a, b \in I^o$ with $a < b$ and $f'' \in L[a,b]$, then the following fractional equality is proved:

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) - \frac{2(1-\alpha)}{\alpha(b-a)} f(k)$$

$$+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_{b}^{\alpha}f)(k)]$$

$$=\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)a)dt$$

$$+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)b)dt.$$

where $\beta(\alpha)>0$ is a normalizer function.

Proof. Let

$$\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)a)dt + \frac{(b-x)^3}{2(b-a)}\int_0^1 t^2 f''(tx+(1-t)b)dt$$

$$=\frac{(x-a)^3}{2(b-a)}I_1+\frac{(b-x)^3}{2(b-a)}I_2.$$

By using the integration by parts, we have

$$I_1=\int_0^1 t^2 f''(tx+(1-t)a)dt$$

$$=\frac{t^2}{x-a}f'(tx+(1-t)a)\Big|_0^1 - \frac{2}{x-a}\int_0^1 tf'(tx+(1-t)a)dt$$

$$=\frac{f'(x)}{x-a}-\frac{2}{x-a}\left[\frac{t}{x-a}f'(tx+(1-t)a)\Big|_0^1 - \frac{1}{x-a}\int_0^1 f(tx+(1-t)a)dt\right]$$

$$=\frac{f'(x)}{x-a}-\frac{2f(x)}{(x-a)^2}+\frac{2}{(x-a)^3}\int_a^x f(u)du. \tag{3.1}$$

Multiplying the equality (??) with $\frac{(x-a)^3}{2}$ and divided by $b-a$ both sides, we get

$$I_1\frac{(x-a)^3}{2(b-a)}$$

$$=\frac{f'(x)(x-a)^3}{2(b-a)(x-a)}-\frac{2f(x)(x-a)^3}{2(x-a)^2(b-a)}+\frac{1}{b-a}\int_a^x f(u)du$$

$$=\frac{f'(x)(x-a)^2}{2(b-a)}-\frac{f(x)(x-a)}{b-a}+\frac{1}{b-a}\int_a^x f(u)du. \tag{3.2}$$

Similarly, we have

$$I_2=\int_0^1 t^2 f''(tx+(1-t)b)dt$$

$$=\frac{-t^2}{b-x}f'(tx+(1-t)b)\Big|_0^1 + \frac{2}{b-x}\int_0^1 tf'(tx+(1-t)b)dt$$

$$=\frac{-f'(x)}{b-x}+\frac{2}{b-x}\left[\frac{-t}{b-x}f''(tx+(1-t)b)\Big|_0^1 + \frac{1}{b-x}\int_0^1 f'(tx+(1-t)b)dt\right]$$

$$=\frac{-f'(x)}{b-x}-\frac{2f(x)}{(b-x)^2}+\frac{2}{(b-x)^3}\int_x^b f(u)du. \tag{3.3}$$

Multiplying the equality (??) with $\frac{(b-x)^3}{2}$ and divided by $(b-a)$ both sides, we get

$$I_2\frac{(b-x)^3}{2(b-a)}=\frac{-f'(x)(b-x)^3}{2(b-a)(b-x)}-\frac{2f(x)(b-x)^3}{(b-x)^2 2(b-a)}+\frac{1}{(b-a)}\int_x^b f(u)du$$

$$=\frac{-f'(x)(b-x)^2}{2(b-a)}-\frac{f(x)(b-x)}{b-a}+\frac{1}{(b-a)}\int_x^b f(u)du. \tag{3.4}$$

Adding the equalities (??) and (??), we get

$$I_1\frac{(x-a)^3}{2(b-a)}+I_2\frac{(b-x)^3}{2(b-a)}$$

$$=\frac{f'(x)(x-a)^2}{2(b-a)}-\frac{f(x)(x-a)}{b-a}+\frac{1}{b-a}\int_a^x f(u)du$$

$$-\frac{f'(x)(b-x)^2}{2(b-a)}-\frac{f(x)(b-x)}{b-a}+\frac{1}{b-a}\int_x^b f(u)du$$

$$=\left(\frac{(x-a)^2}{2(b-a)}-\frac{(b-x)^2}{2(b-a)}\right)f'(x)$$

$$-f(x)\left(\frac{x-a}{b-a}+\frac{b-x}{b-a}\right)+\frac{1}{(b-a)}\int_a^b f(u)du. \tag{3.5}$$

Adding $\frac{2(1-\alpha)}{\beta(\alpha)(b-a)}f(k)$ both sides the equality (??), we have

$$I_1\frac{(x-a)^3}{2(b-a)}+I_2\frac{(b-x)^3}{2(b-a)}+\frac{2(1-\alpha)}{\beta(\alpha)(b-a)}f(k)$$

$$=\left(\frac{(x-a)^2}{2(b-a)}-\frac{(b-x)^2}{2(b-a)}\right)f'(x)-f(x)\left(\frac{x-a}{b-a}+\frac{b-x}{b-a}\right)$$

$$+\frac{\beta(\alpha)}{\alpha(b-a)}\left(\frac{\alpha}{\beta(\alpha)}\int_a^k f(u)du+\frac{1-\alpha}{\beta(\alpha)}f(k)\right.$$
$$\left.+\frac{\alpha}{\beta(\alpha)}\int_k^b f(u)du+\frac{1-\alpha}{\beta(\alpha)}f(k)\right)$$

$$=\left(\frac{(x-a)^2}{2(b-a)}-\frac{(b-x)^2}{2(b-a)}\right)f^{'}(x)-f(x)\left(\frac{x-a}{b-a}+\frac{b-x}{b-a}\right)$$

$$+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)].$$

Thus, we have

$$\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)$$
$$+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)$$
$$+(CFI_b^{\alpha}f)(k)]$$

$$=\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2f^{''}(tx+(1-t)a)dt$$
$$+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2f^{''}(tx+(1-t)b)dt.$$

The proof of Lemma 1 is completed.

Theorem 1 Given the validity of Lemma 1. If $|f''|$ is trigonometrically convex$whitei$on $[a,b]$, then the$whitei$following fractional inequality is established:

$$\left|\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right.$$

$$\left.+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]\right|$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\frac{8\pi-16}{\pi^3}\left|f^{''}(x)\right|+\frac{2\pi^2-16}{\pi^3}\left|f^{''}(a)\right|\right]$$

$$+\frac{(b-x)^3}{2(b-a)}\left[\frac{8\pi-16}{\pi^3}\left|f^{''}(x)\right|+\frac{2\pi^2-16}{\pi^3}\left|f^{''}(b)\right|\right].$$

Proof. By using the$whitei$Lemma 1, since $|f''|$ is trigonometrically convex, we$whitei$have

$$\left|\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right.$$

$$\left.+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]\right|$$
$$\leq\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left|f^{''}(tx+(1-t)a)\right|dt$$
$$+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left|f^{''}(tx+(1-t)b)\right|dt$$

$$\leq\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left(sin\left(\frac{\pi t}{2}\right)\left|f^{''}(x)\right|\right.$$
$$\left.+cos\left(\frac{\pi t}{2}\right)\left|f^{''}(a)\right|\right)dt$$
$$+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left(sin\left(\frac{\pi t}{2}\right)\left|f^{''}(x)\right|\right.$$
$$\left.+cos\left(\frac{\pi t}{2}\right)\left|f^{''}(b)\right|\right)dt$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left(\int_0^1 t^2\,sin\left(\frac{\pi t}{2}\right)\left|f^{''}(x)\right|\right.$$
$$\left.+\int_0^1 t^2\,cos\left(\frac{\pi t}{2}\right)\left|f^{''}(a)\right|\right)dt$$
$$+\frac{(b-x)^3}{2(b-a)}\left(\int_0^1 t^2\,sin\left(\frac{\pi t}{2}\right)\left|f^{''}(x)\right|\right.$$
$$\left.+\int_0^1 t^2\,cos\left(\frac{\pi t}{2}\right)\left|f^{''}(b)\right|\right)dt$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\frac{8\pi-16}{\pi^3}\left|f^{''}(x)\right|+\frac{2\pi^2-16}{\pi^3}\left|f^{''}(a)\right|\right]$$
$$+\frac{(b-x)^3}{2(b-a)}\left[\frac{8\pi-16}{\pi^3}\left|f^{''}(x)\right|+\frac{2\pi^2-16}{\pi^3}\left|f^{''}(b)\right|\right].$$

The proof of Theorem 1, is completed.

Corollary 1 If we choose $x=\frac{a+b}{2}$ in Theorem 1, then we get

$$\left|\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-f\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{4}\left(\frac{\pi^2-8}{\pi^3}A\left(\left|f^{''}(a)\right|+\left|f^{''}(b)\right|\right)\right.$$
$$\left.+\frac{4\pi-8}{\pi^3}\left|f^{''}\left(\frac{a+b}{2}\right)\right|\right).$$

Theorem 2 Under the assumptions of Lemma 1 holds. If $|f''|^q$is trigonometrically convex$whitei$on $[a,b]$ and $q>1$, then$whitei$the$whitei$following fractional inequality is proved:

$$\left|\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right.$$

$$\left.+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_a^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]\right|$$

$$\leq\left[\frac{(x-a)^3}{2(b-a)}\left(\left|f^{''}(x)\right|^q+\left|f^{''}(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^3}{2(b-a)}\left(\left|f^{''}(x)\right|^q+\left|f^{''}(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\times\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}.$$

Proof. By using the Lemma 1, with the help of Hölder inequality and trigonometrically convexity of $|f''|^q$, we have

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]\right|$$

$$\leq\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2|f''(tx+(1-t)a)|dt+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2|f''(tx+(1-t)b)|dt$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)a)dt|^q dt\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-x)^3}{2(b-a)}\left[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)b)dt|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(sin\left(\frac{\pi t}{2}\right)|f''(x)|^q+cos\left(\frac{\pi t}{2}\right)|f''(a)|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-x)^3}{2(b-a)}\left[\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(sin\left(\frac{\pi t}{2}\right)|f''(x)|^q+cos\left(\frac{\pi t}{2}\right)|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$\leq\left[\frac{(x-a)^3}{2(b-a)}\left(|f''(x)|^q+|f''(a)|^q\right)^{\frac{1}{q}}+\frac{(b-x)^3}{2(b-a)}\left(|f''(x)|^q+|f''(b)|^q\right)^{\frac{1}{q}}\right]\times\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}.$$

The proof of Theorem 2, is finished.

Corollary 2 If we choose $x=\frac{a+b}{2}$ and $p=q=2$ in Theorem 2, then we get

$$\left|\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{16}\left(\frac{2}{5\pi}\right)^{\frac{1}{2}}\times\left[\left(\left|f''\left(\frac{a+b}{2}\right)\right|^2+|f''(a)|^2\right)^{\frac{1}{2}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^2+|f''(b)|^2\right)^{\frac{1}{2}}\right].$$

Corollary 3 If we choose $f''(a)=f''(b)=f''\left(\frac{a+b}{2}\right)$ in Corollary 2, then we get

$$\left|\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{8}\left(\frac{1}{5\pi}\right)^{\frac{1}{2}}\left|f''\left(\frac{a+b}{2}\right)\right|.$$

Theorem 3 Under the of Lemma 1 holds. If $|f''|^q$is trigonometrically convex$whitei$on $[a,b]$ and $q\geq 1$, then$whitei$the$whitei$following fractional inequality is constructed:

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]\right|$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\frac{8\pi-16}{\pi^3}|f''(x)|^q+\frac{2\pi^2-16}{\pi^3}|f''(a)|^q\right]^{\frac{1}{q}}$$

$$+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\frac{8\pi-16}{\pi^3}|f''(x)|^q+\frac{2\pi^2-16}{\pi^3}|f''(b)|^q\right]^{\frac{1}{q}}.$$

Proof. By using $whitei$ the Lemma 1, with the help of $whitei$ power-mean inequality $whitei$ and trigonometrically convexity of $|f''|^q$, we$whitei$have

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)+\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_{b}^{\alpha}f)(k)]\right|$$

$$\leq\frac{(x-a)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)a)\right|dt+\frac{(b-x)^3}{2(b-a)}\int_0^1 t^2\left|f''(tx+(1-t)b)\right|dt$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f''(tx+(1-t)a)dt\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-x)^3}{2(b-a)}\left[\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f''(tx+(1-t)b)dt\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left[\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left(sin\left(\frac{\pi t}{2}\right)\left|f''(x)\right|^q+cos\left(\frac{\pi t}{2}\right)\left|f''(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-x)^3}{2(b-a)}\left[\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left(sin\left(\frac{\pi t}{2}\right)\left|f''(x)\right|^q+cos\left(\frac{\pi t}{2}\right)\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(x-a)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\frac{8\pi-16}{\pi^3}\left|f''(x)\right|^q+\frac{2\pi^2-16}{\pi^3}\left|f''(a)\right|^q\right]^{\frac{1}{q}}$$

$$+\frac{(b-x)^3}{2(b-a)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\frac{8\pi-16}{\pi^3}\left|f''(x)\right|^q+\frac{2\pi^2-16}{\pi^3}\left|f''(b)\right|^q\right]^{\frac{1}{q}}.$$

This $whitei$ completes $whitei$ the proof.

Corollary 4 If $whitei$ we choose $q=1$ in $whitei$ Theorem 3, then we get Theorem 1.

Corollary 5 If $whitei$ we choose $x=\frac{a+b}{2}$ and $p=q=2$ in $whitei$ Theorem 3, then we get

$$\left|\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_{b}^{\alpha}f)(k)]-f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{2}}\left[\left(\frac{8\pi-16}{\pi^3}\left|f''\left(\frac{a+b}{2}\right)\right|^2+\frac{2\pi^2-16}{\pi^3}\left|f''(a)\right|^2\right)^{\frac{1}{2}}+\left(\frac{8\pi-16}{\pi^3}\left|f''\left(\frac{a+b}{2}\right)\right|^2+\frac{2\pi^2-16}{\pi^3}\left|f''(b)\right|^2\right)^{\frac{1}{2}}\right].$$

Corollary 6 If we choose $f''(a)=f''(b)=f''\left(\frac{a+b}{2}\right)$ in Theorem 3, then we get

$$\left|\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_{b}^{\alpha}f)(k)]-f\left(\frac{a+b}{2}\right)-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{8}\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(\frac{\pi^2+4\pi-16}{\pi^3}\right)\left|f''\left(\frac{a+b}{2}\right)\right|.$$

4. TRAPEZOID-TYPE INEQUALITIES FOR EXPONENTIAL TRIGONOMETRIC-CONVEX FUNCTIONS

In this section, trapezoid-type inequalities are established for exponential trigonometric convex functions by means of Caputo-Fabrizio integrals. To develop these inequalities first, let's create the following lemma:

Lemma 2 Suppose $I\subseteq R$, $f: I\to R$ be a twice differentiable mapping on I^o where $a,b\in I^o$ with $a<b$ and $f''\in L[a,b]$, then the following fractional equality is proved:

$$\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_{b}^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 (t-t^2)f''(ta+(1-t)b)dt.$$

Proof. By using the integration by parts, we have

$$I=\int_0^1 (t-t^2)f''(ta+(1-t)b)dt$$

$$=\frac{-(t-t^2)}{b-a}f'(ta+(1-t)b)\Big|_0^1+\frac{1}{b-a}\int_0^1 (1-2t)f'(ta+(1-t)b)dt$$

$$=\frac{1}{b-a}\int_0^1 (1-2t)f'(ta+(1-t)b)dt$$

$$=\frac{1}{b-a}\left[\frac{1-2t}{b-a}f(ta+(1-t)b)\Big|_0^1-\frac{2}{b-a}\int_0^1 f(ta+(1-t)b)dt\right]$$

$$=\frac{f(a)+f(b)}{(b-a)^2}-\frac{2}{(b-a)^2}\int_0^1 f(ta+(1-t)b)dt$$

$$=\frac{f(a)+f(b)}{(b-a)^2}-\frac{2}{(b-a)^3}\int_a^b f(u)du. \quad (4.1)$$

Multiplying the equality (??) with $\frac{(b-a)^2}{2}$ and adding $\frac{2(1-\alpha)}{\beta(\alpha)(b-a)}f(k)$ both sides, we get

$$I\frac{(b-a)^2}{2}+\frac{2(1-\alpha)}{\beta(\alpha)(b-a)}f(k)$$

$$=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du+\frac{2(1-\alpha)}{\beta(\alpha)(b-a)}f(k)$$

$$=\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}\times\left(\frac{\alpha}{\beta(\alpha)}\int_a^k f(u)du-\frac{(1-\alpha)}{\beta(\alpha)}f(k)+\frac{\alpha}{\beta(\alpha)}\int_k^b f(u)du-\frac{(1-\alpha)}{\beta(\alpha)}f(k)\right)$$

$$=\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)].$$

Thus, we have

$$\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 (t-t^2)f''(ta+(1-t)b)dt.$$

The proof of Lemma 2 is completed.

Theorem 4 Given the validity of Lemma 2. If $|f''|$ is exponential trigonometric convex $whitei$ on $[a,b]$, then the $whitei$ following fractional inequality is established:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{96\pi^2-25\pi e^{-1}-4\pi^4-64}{(\pi^2+4)^3}\right)A\left(|f''(a)|+|f''(b)|\right).$$

Proof. By using $whitei$ the $whitei$ Lemma 2, we $whitei$ have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 |t-t^2|\left(\frac{sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|+\frac{cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|\right)dt$$

$$\leq\frac{(b-a)^2}{2}\left[\left(\frac{96\pi^2-25\pi e^{-1}-4\pi^4-64}{(\pi^2+4)^3}\right)|f''(a)|+\left(\frac{96\pi^2-25\pi e^{-1}-4\pi^4-64}{(\pi^2+4)^3}\right)|f''(b)|\right]$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{96\pi^2-25\pi e^{-1}-4\pi^4-64}{(\pi^2+4)^3}\right)A\left(|f''(a)|+|f''(b)|\right).$$

This $whitei$ completes $whitei$ the proof.

Theorem 5 Under the assumptions of Lemma 2 holds. If $|f''|^q$is exponential trigonometric $whitei$ convex on $[a,b]$ and $q>1$, then $whitei$ the $whitei$f ollowing fractional inequality is proved:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{4\pi e^{-1}+8}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right).$$

Proof. By using the Lemma 2, with the help of Hölder inequality and exponential trigonometric convexity of $|f''|^q$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt$$

$$\leq\frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p\right)^{\frac{1}{p}}\left(|f''(ta+(1-t)b)|^q\right)^{\frac{1}{q}}dt$$

$$\leq\frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)^p\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(\frac{sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{2\pi e^{-1}+4}{\pi^2+4}|f''(a)|^q+\frac{2\pi e^{-1}+4}{\pi^2+4}|f''(b)|^q\right)^{\frac{1}{q}}$$

$$\leq\frac{(b-a)^2}{2}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{4\pi e^{-1}+8}{\pi^2+4}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right).$$

This *whitei* completes *whitei* the proof.

Theorem 6 Under the assumptions of Lemma 2 holds. If $|f''|^q$is exponential trigonometric convex *whitei* on $[a,b]$ and $q\geq 1$, then *whitei* the *whitei* following fractional inequality is constructed:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{-4(\pi^2+4)^2-128\pi(2e^{-1}-\pi)}{(\pi^2+4)^3}\right)A^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right).$$

Proof. By using *whitei* the Lemma 2, with the help of *whitei* power-mean inequality *whitei* and exponential trigonometric convex of $|f''|^q$, we *whitei* have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\beta(\alpha)}{\alpha(b-a)}[({}_{a}^{CF}I^{\alpha}f)(k)+(CFI_b^{\alpha}f)(k)]-\frac{2(1-\alpha)}{\alpha(b-a)}f(k)\right|$$

$$\leq\frac{(b-a)^2}{2}\int_0^1 |t-t^2||f''(ta+(1-t)b)|dt$$

$$\leq\frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)\right)^{1-\frac{1}{q}}\left(\int_0^1 (t-t^2)|f''(ta+(1-t)b)dt|^q dt\right)^{\frac{1}{q}}dt$$

$$\leq\frac{(b-a)^2}{2}\left(\int_0^1 (t-t^2)\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 (t-t^2)\left(\frac{sin\left(\frac{\pi t}{2}\right)}{e^{1-t}}|f''(a)|^q+\frac{cos\left(\frac{\pi t}{2}\right)}{e^{t}}|f''(b)|^q\right)dt\right)^{\frac{1}{q}}$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{6}}\left(\left(\frac{-4(\pi^2+4)^2-128\pi(2e^{-1}-\pi)}{(\pi^2+4)^3}\right)|f''(a)|^q+\left(\frac{-4(\pi^2+4)^2-128\pi(2e^{-1}-\pi)}{(\pi^2+4)^3}\right)|f''(b)|^q\right)^{\frac{1}{q}}$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{-4(\pi^2+4)^2-128\pi(2e^{-1}-\pi)}{(\pi^2+4)^3}\right)A^{\frac{1}{q}}\left(|f''(a)|^q+|f''(b)|^q\right).$$

This *whitei* completes *whitei* the proof.

5. CONCLUSION

Novel error bounds and estimates of Ostrowski-type and trapezoidal-type inequalities have been established for Caputo-Fabrizio fractional integrals under twice-differentiable functions. These inequalities have been further applied to trigonometric and exponential trigonometric convex functions, broadening their scope of applicability.

It is anticipated that future studies in fractional calculus will extend similar Ostrowski-type and trapezoidal-type inequalities to other fractional operators, such as conformable and tempered fractional integrals, as well as coordinated preinvex functions. Furthermore, the exploration of these derived inequalities through various classes of convexity is suggested as a potential avenue for research.

CONFLICT OF INTEREST

Not Applicable.

FUNDING

Not Applicable.

ETHICAL APPROVAL

Not Applicable.

INFORMED CONSENT

Not Applicable.

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