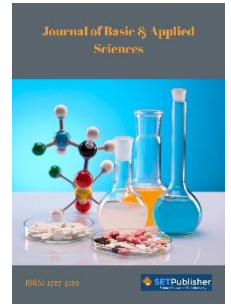




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Soft Intersection Quasi-interior Ideals of Semigroups

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Abstract:

It has been shown that generalizing the ideals of an algebraic structure is both interesting and beneficial for mathematicians. In this context, the concept of quasi-interior (QI) ideal was introduced as a generalization of quasi-ideal and interior ideal of a semigroup. In this paper, we apply this concept to soft set theory and semigroups, introducing a new form of soft intersection (S-int) ideal called the "soft intersection (S-int) quasi-interior (QI) ideal." The main objective of this study is to investigate the relationships between S-int QI ideals and other specific types of S-int ideals in a semigroup. It has been shown that every S-int interior ideal of a semigroup is an S-int QI ideal, and every S-int ideal is an S-int QI ideal. The S-int bi-ideal of a group is an S-int QI ideal, the S-int quasi-ideal of a regular group is an S-int QI ideal, the idempotent S-int QI ideal is an S-int bi-quasi-ideal and an S-int bi-interior ideal. Counterexamples are provided to show that the opposites of these statements are not always valid. We prove that for the converses to hold, the semigroup should be a group or regular, or the S-int QI ideal should be idempotent. Our main theorem, which demonstrates that if a subsemigroup of a semigroup is a QI ideal, then its soft characteristic function is an S-int QI ideal, and vice versa, enables us to establish a connection between semigroup theory and soft set theory. Through this theorem, we illustrate how this concept connects to the existing algebraic structures in classical semigroup theory. Additionally, we offer conceptual characterizations and an analysis of the concept in terms of soft set operations, including soft image and soft inverse image, supporting our claims with specific, informative examples. Furthermore, the connection between a regular semigroup and the structure of S-int QI ideals is established and presented.

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1. INTRODUCTION

Semigroups play a key role in various areas of mathematics as they provide the abstract algebraic foundation for "memoryless" systems, which reset with each iteration. In applied mathematics, semigroups—first formally studied in the early 1900s—are crucial models for linear time-invariant systems. Since finite semigroups are closely connected to finite automata, their study is vital for theoretical computer science. Moreover, in probability theory, there is a connection between semigroups and Markov processes. The concept of ideals is essential for understanding mathematical structures and their applications. As a result, many mathematicians have concentrated much of their research on generalizing ideals within algebraic structures. In other words, further exploration of algebraic structures necessitates the generalization of ideals within these structures. By utilizing the concept and properties of ideal generalizations in algebraic structures, several mathematicians have made significant discoveries and characterizations of these structures. Dedekind introduced the concept of ideals in the theory of algebraic numbers, and Noether extended it to include associative rings. The notion of a one-sided ideal in any algebraic structure is an extension of the ideal concept, and both one-sided and two-sided ideals remain fundamental concepts in ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals for semigroups. Steinfeld [2] was the first to present the notion of quasi-ideals for semigroups and later for rings. Quasi-ideal is the generalization of right and left ideal, while bi-ideal is a further generalization of quasi-ideal. The concept of interior ideals was first introduced by Lajos [3] and later studied in more detail by Szasz [4,5]. Interior ideals are a generalization of ideals. Rao [6-9] developed several new types of ideals for semigroups, which are generalizations of existing ones, such as bi-interior ideals, bi-quasi-ideals, $Q|$ ideals, weak-interior ideals, and bi-quasi-interior ideals. Furthermore, the concept of essential ideals in semigroups was introduced by Baupradist *et al.* [10]. As a more generalized idea of the various types of ideals, the concept of "almost" ideals was proposed, and their characteristics, as well as their relationships with related ideals, were thoroughly examined. In this context, the concept of almost ideals of semigroups was first introduced in [11]. A subsequent paper [12] extended the concept of bi-ideals to almost bi-ideals of semigroups. While the notion of almost quasi-ideals was first introduced in [13], the concepts of almost

interior ideals and weakly almost interior ideals of semigroups were proposed, expanding and studying the ideas of almost ideals and interior ideals of semigroups in [14]. Almost subsemigroups, almost bi-quasi-ideals, almost bi-interior ideals, and almost bi-quasi-interior ideals of semigroups were introduced by the authors in [15–18], respectively. Furthermore, in [13, 15–20], several fuzzy almost ideal types for semigroups were explored.

Molodtsov [21] introduced "Soft Set Theory" in 1999 to address problems involving uncertainty and to find appropriate solutions for them. Since then, numerous important studies have been carried out on concepts related to soft sets, particularly concerning soft set operations. Maji *et al.* [22] provided some definitions related to soft sets and defined specific operations on soft sets. Pei and Miao [23] and Ali *et al.* [24] introduced various operations on soft sets. Sezgin and Atagün [25] worked on soft set operations as well. For more information on soft set operations, which have gained popularity since their inception, we refer to [26-37]. The concept and operations of soft sets were modified by Çağman and Enginoğlu [38]. Çağman *et al.* [39] introduced the concept of soft int-groups, which then prompted the exploration of various soft algebraic systems. Sezer *et al.* [40,41], by applying soft sets to semigroup theory, introduced the concepts of soft intersection (S-int) semigroups, left (right /two-sided) ideals, interior ideals, quasi-ideals, and (generalized) bi-ideals of semigroups, and extensively studied their fundamental properties. Regarding the S-int substructures of semigroups, Sezgin and Orbay [42] defined and classified semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, the semi-lattices of left (right) simple semigroups, the semi-lattices of left (right) groups, and the semi-lattices of groups. S-int almost ideals were introduced and explored as a generalization of various types of S-int ideals in references [43-54]. The soft versions of different algebraic structures have been investigated in references [55-67].

Rao [68] defined and studied $Q|$ ideals and fuzzy $Q|$ ideals in semirings and explored their relationships. Rao [69] also examined the structure of $Q|$ ideals in semirings. Rao [70, 71] defined and explored the relationships and properties of $Q|$ ideals in semirings, $Q|$ ideals in Γ -semirings, and fuzzy $Q|$ ideals. Rao [9, 72] introduced the concepts of $Q|$ ideals and weak-interior ideals in semigroups, and defined the structures of $Q|$ ideals and fuzzy $Q|$ ideals in semigroups. Rao

[73] studied fuzzy soft Q| ideals in Γ -semirings and investigated their properties. Saeid, Rao, Kona, and Rafi [74] introduced the concept of fuzzy (soft) Q| ideals in semirings and characterized regular semirings in terms of their fuzzy (soft) Q| ideals. Rao [75] defined and exemplified Q| ideals and their properties in Γ -semirings, presenting them with theorems and proofs. Srikanth and Shobhalatha [76] studied Q| ideals and fuzzy soft Q| ideals in triple semirings and explored their properties. Rao [77] introduced the Q| ideals and fuzzy Q| ideals of Γ -semigroups into the literature. Thus, the structure of Q| ideals and their relationships in classical and fuzzy algebra have been studied.

In this paper, we apply this concept to soft set theory and semigroups by introducing "S-int Q| ideals of semigroups." We explore the relationships between S-int Q| ideals and other types of S-int ideals in a semigroup. It has been shown that every S-int interior ideal of a semigroup is an S-int Q| ideal, every S-int ideal is an S-int Q| ideal, the S-int bi-ideal of a group is an S-int Q| ideal, the S-int quasi-ideal of a regular group is an S-int Q| ideal, the idempotent S-int Q| ideal is an S-int bi-quasi-ideal and an S-int bi-interior ideal. Counterexamples are provided to show that the opposites of these are not always valid. It has also been proven that the validity of the opposites depends on additional conditions. It has been obtained that the S-int interior ideal of a group is an S-int Q| ideal, for every idempotent soft sets, S-int Q| ideal is an S-int ideal, the S-int ideal of a regular semigroup is an S-int Q| ideal, the S-int bi-ideal of a group is an S-int Q| ideal, for every idempotent soft sets, S-int Q| ideal is an S-int bi-ideal, and the S-int quasi-ideal of a regular group is an S-int Q| ideal. Our key theorem, which asserts that if a subsemigroup of a semigroup is a Q| ideal, its soft characteristic function is an S-int Q| ideal, and vice versa, allows us to bridge the gap between semigroup theory and soft set theory. Using this theorem, we illustrate how this concept relates to the existing algebraic structures in classical semigroup theory. The characterizations of S-int Q| ideals under soft set operations, such as soft image and inverse image, are also provided. Additionally, the relationship between a regular semigroup and the structure of S-int Q| ideals is proven and presented. Section 1 offers a general overview of the topic, while Section 2 explores the fundamental concepts of semigroups and soft set ideals, along with their relevant definitions and implications. In Section 3, we introduce the concept of S-int Q| ideals and examine its properties, as well as its relationships with other types of S-int ideals through

concrete examples. Section 4 provides a summary of our findings and discusses potential avenues for future research.

2. PRELIMINARIES

Throughout this paper, S denotes a semigroup. A nonempty subset C of S is called a subsemigroup of S if $CC \subseteq C$, is called a bi-ideal of S if $CC \subseteq C$ and $CSC \subseteq C$, is called an interior ideal of S if $SCS \subseteq C$, and is called a quasi-ideal of S if $CS \cap SC \subseteq C$. A subsemigroup C of S is called a left (L-) Q| ideal of S if $SCSC \subseteq C$, is called a right (R-) Q| ideal of S if $CSCS \subseteq C$, and is called a Q| ideal of S if it is both L-Q| ideal of S and R-Q| ideal of S [9]. If S is a regular semigroup, then for all $\eta \in S$, there exists an element $z \in S$ such that $\eta = \eta z \eta$. A semigroup S is called L-simple (R-simple) if it contains no proper L- (R-) ideal of S and is called simple if it contains no proper ideal.

Theorem 2.1 [78, 79]. Let S be a semigroup. Then,

1. S is L-simple (R-simple) if and only if (iff) $Su = S(uS = S)$ for all $u \in S$. That is, for every $u, b \in S$, there exists $m \in S$ such that $b = mu$ ($b = um$)
2. S is both L-simple and R-simple iff S is a group.

Definition 2.2 [21]. Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U , and $\beta \subseteq E$. The soft set (SS) \tilde{f}_β over U is a function such that $\tilde{f}_\beta: E \rightarrow P(U)$, where for all $u \notin \beta$, $\tilde{f}_\beta(u) = \emptyset$. That is,

$$\tilde{f}_\beta = \{(u, \tilde{f}_\beta(u)): u \in E, \tilde{f}_\beta(u) \in P(U)\}$$

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.3 [38]. Let $\tilde{f}_\beta \in S_E(U)$. If $\tilde{f}_\beta(\xi) = \emptyset$ for all $\xi \in E$, then \tilde{f}_β is called a null SS and indicated by \emptyset_E .

Definition 2.4 [38]. Let $\tilde{f}_c, \tilde{f}_d \in S_E(U)$. If $\tilde{f}_c(a) \subseteq \tilde{f}_d(a)$, for all $a \in E$, then \tilde{f}_c is a soft subset of \tilde{f}_d and indicated by $\tilde{f}_c \subseteq \tilde{f}_d$. If $\tilde{f}_c(a) = \tilde{f}_d(a)$, for all $a \in E$, then \tilde{f}_c is called soft equal to \tilde{f}_d and denoted by $\tilde{f}_c = \tilde{f}_d$.

Definition 2.5 [38]. Let $\tilde{f}_c, \tilde{f}_d \in S_E(U)$. The union (intersection) of \tilde{f}_c and \tilde{f}_d is the SS $\tilde{f}_c \cup \tilde{f}_d$ ($\tilde{f}_c \cap \tilde{f}_d$), where $(\tilde{f}_c \cup \tilde{f}_d)(r) = \tilde{f}_c(r) \cup \tilde{f}_d(r)$ ($(\tilde{f}_c \cap \tilde{f}_d)(r) = \tilde{f}_c(r) \cap \tilde{f}_d(r)$), for all $r \in E$, respectively.

Definition 2.6 [38]. Let $\tilde{f}_c, \tilde{f}_d \in S_E(U)$. Then, \wedge -product (\vee -product) of \tilde{f}_c and \tilde{f}_d , denoted by $\tilde{f}_c \wedge \tilde{f}_d$ ($\tilde{f}_c \vee \tilde{f}_d$) is defined by $(\tilde{f}_c \wedge \tilde{f}_d)(r, z) = \tilde{f}_c(r) \cap \tilde{f}_d(z)$ ($(\tilde{f}_c \vee \tilde{f}_d)(r, z) = \tilde{f}_c(r) \cup \tilde{f}_d(z)$) for all $(r, z) \in E \times E$, respectively.

Definition 2.7 [39]. Let $\tilde{f}_\mathfrak{R}, \tilde{f}_\mathfrak{U} \in S_E(U)$, and ζ , be a function from \mathfrak{R} to \mathfrak{U} . Then, soft image of $\tilde{f}_\mathfrak{R}$ under ζ , and soft pre-image (or soft inverse image) of $\tilde{f}_\mathfrak{U}$ under ζ

are the SSs $\zeta(f_{\mathfrak{R}})$ and $\zeta^{-1}(f_{\mathfrak{U}})$ such that

$$(\zeta(f_{\mathfrak{R}}))(\zeta) = \begin{cases} \bigcup \{f_{\mathfrak{R}}(\check{v}) \mid \check{v} \in \mathfrak{R} \text{ and } \zeta(\check{v}) = \zeta\}, & \text{if } \zeta^{-1}(\zeta) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $\zeta \in \mathfrak{U}$ and $(\zeta^{-1}(f_{\mathfrak{U}}))(\check{v}) = f_{\mathfrak{U}}(\zeta(\check{v}))$ for all $\check{v} \in \mathfrak{R}$.

Definition 2.8 [39]. Let $f_{\mathfrak{B}} \in S_E(U)$ and $\mathfrak{N} \subseteq U$. Then, upper \mathfrak{N} -inclusion of $f_{\mathfrak{B}}$, denoted by $\mathcal{U}(f_{\mathfrak{B}}; \mathfrak{N})$, is defined as

$$\mathcal{U}(f_{\mathfrak{B}}; \mathfrak{N}) = \{x \in \mathfrak{B} \mid f_{\mathfrak{B}}(x) \supseteq \mathfrak{N}\}$$

Definition 2.9 [40]. Let $f_S, y_S \in S_S(U)$. S-int product $f_S \circ y_S$ is defined by

$$(f_S \circ y_S)(d) = \begin{cases} \bigcup_{d=xz} \{f_S(x) \cap y_S(z)\}, & \text{if } \exists x, z \in S \text{ such that } d = xz \\ \emptyset, & \text{otherwise} \end{cases}$$

Theorem 2.10 [40]. Let S be a semigroup. Then,

$$(f_S \circ y_S) \circ \rho_S = f_S \circ (y_S \circ \rho_S) \\ f_S \circ y_S \neq y_S \circ f_S$$

$$f_S \circ (y_S \tilde{\cup} \rho_S) = (f_S \circ y_S) \tilde{\cup} (f_S \circ \rho_S) \text{ and } (f_S \tilde{\cup} y_S) \circ \rho_S = (f_S \circ \rho_S) \tilde{\cup} (y_S \circ \rho_S)$$

$$f_S \circ (y_S \tilde{\cap} \rho_S) = (f_S \circ y_S) \tilde{\cap} (f_S \circ \rho_S) \text{ and } (f_S \tilde{\cap} y_S) \circ \rho_S = (f_S \circ \rho_S) \tilde{\cap} (y_S \circ \rho_S)$$

If $f_S \subseteq y_S$, then $f_S \circ \rho_S \subseteq y_S \circ \rho_S$ and $\rho_S \circ f_S \subseteq \rho_S \circ y_S$

If $n_S, h_S \in S_S(U)$ such that $n_S \subseteq f_S$ and $h_S \subseteq y_S$, then $n_S \circ h_S \subseteq f_S \circ y_S$.

Definition 2.11 [40]. Let $\emptyset \neq \mathfrak{B} \subseteq S$. The soft characteristic function ($\square CF$) of \mathfrak{B} , denoted by $S_{\mathfrak{B}}$, is defined as

$$S_{\mathfrak{B}}(z) = \begin{cases} U, & \text{if } z \in \mathfrak{B} \\ \emptyset, & \text{if } z \in S \setminus \mathfrak{B} \end{cases}$$

Theorem 2.12 [40, 49]. Let $\mathfrak{B}, \mathfrak{R} \subseteq S$. Then,

$$\mathfrak{B} \subseteq \mathfrak{R} \text{ iff } S_{\mathfrak{B}} \subseteq S_{\mathfrak{R}}$$

$$S_{\mathfrak{B}} \tilde{\cap} S_{\mathfrak{R}} = S_{\mathfrak{B} \cap \mathfrak{R}} \text{ and } S_{\mathfrak{B}} \tilde{\cup} S_{\mathfrak{R}} = S_{\mathfrak{B} \cup \mathfrak{R}}$$

$$S_{\mathfrak{B}} \circ S_{\mathfrak{R}} = S_{\mathfrak{B}\mathfrak{R}}$$

Definition 2.13 [40]. A SS l_S over U is called an S-int subsemigroup of S if $l_S(e\mathfrak{C}) \supseteq l_S(e) \cap l_S(\mathfrak{C})$ for all $e, \mathfrak{C} \in S$.

Here note that in [40], the definition of ‘‘S-int subsemigroup of S ’’ is given as ‘‘S-int semigroup of S ’’; however in this paper, without loss of generality, we prefer to use ‘‘S-int subsemigroup of S ’’.

Definition 2.14 [40, 41]. A SS l_S over U is called an S-int L- (R-) ideal of S if $l_S(m\mathfrak{S}) \supseteq l_S(\mathfrak{S})$ ($l_S(m\mathfrak{S}) \supseteq l_S(\mathfrak{M})$) for all $m, \mathfrak{S} \in S$, and is called an S-int two-sided ideal (S-int ideal) of S if it is both S-int L-ideal of S over U and S-int R-ideal of S over U . An S-int subsemigroup l_S is called an S-int bi-ideal of S if $l_S(\mathfrak{d}\mathfrak{p}\check{u}) \supseteq l_S(\mathfrak{d}) \cap l_S(\check{u})$ for all $\mathfrak{d}, \mathfrak{p}, \check{u} \in S$. A SS l_S over U is called an S-int interior ideal of S if $l_S(\mathfrak{j}\mathfrak{s}\mathfrak{C}) \supseteq l_S(\mathfrak{s})$ for all $\mathfrak{j}, \mathfrak{C}, \mathfrak{s} \in S$.

It is easy to see that if $l_S(\alpha) = U$ for all $\alpha \in S$, then l_S is an S-int subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal). We denote such a kind of S-int subsemigroup (L-ideal, R-ideal, ideal, bi-ideal, interior ideal) by $\tilde{\mathfrak{S}}$. It is obvious that $\tilde{\mathfrak{S}} = S_S$, that is, $\tilde{\mathfrak{S}}(\alpha) = U$ for all $\alpha \in S$ [40, 41].

Definition 2.15 [41, 80, 81]. A SS l_S over U is called an S-int quasi-ideal of S over U if $(l_S \circ \tilde{\mathfrak{S}}) \tilde{\cap} (\tilde{\mathfrak{S}} \circ l_S) \subseteq l_S$. A SS l_S over U is called an S-int L- (R-) bi-quasi ideal of S if $(\tilde{\mathfrak{S}} \circ l_S) \tilde{\cap} (l_S \circ \tilde{\mathfrak{S}}) \subseteq l_S$ ($(l_S \circ \tilde{\mathfrak{S}}) \tilde{\cap} (l_S \circ \tilde{\mathfrak{S}}) \subseteq l_S$) and is called an S-int bi-quasi ideal of S if it is both S-int L-bi-quasi ideal of S over U and S-int R-bi-quasi ideal of S over U . A SS l_S over U is called an S-int bi-interior ideal of S over U if $(\tilde{\mathfrak{S}} \circ l_S \circ \tilde{\mathfrak{S}}) \tilde{\cap} (l_S \circ \tilde{\mathfrak{S}} \circ l_S) \subseteq l_S$.

Theorem 2.16 [40]. Let $g_S \in S_S(U)$. Then,

- i) $\tilde{\mathfrak{S}} \circ \tilde{\mathfrak{S}} \subseteq \tilde{\mathfrak{S}}$
- ii) $\tilde{\mathfrak{S}} \circ g_S \subseteq \tilde{\mathfrak{S}}$ and $g_S \circ \tilde{\mathfrak{S}} \subseteq \tilde{\mathfrak{S}}$
- iii) $g_S \tilde{\cup} \tilde{\mathfrak{S}} = \tilde{\mathfrak{S}}$ and $g_S \tilde{\cap} \tilde{\mathfrak{S}} = g_S$.

Theorem 2.17 [40, 41]. Let \mathfrak{F} be a nonempty subset of S . Then, \mathfrak{F} is a subsemigroup (L-ideal, R-ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, bi-quasi ideal, bi-interior ideal) of S iff $S_{\mathfrak{F}}$ is an S-int subsemigroup (L-ideal, R-ideal, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, bi-quasi ideal, bi-interior ideal).

Theorem 2.18 [40, 41]. Let $l_S \in S_S(U)$. Then, l_S is an S-int subsemigroup $\Leftrightarrow (l_S \circ l_S) \subseteq l_S$, l_S is an S-int L- (R-) ideal $\Leftrightarrow (\tilde{\mathfrak{S}} \circ l_S) \subseteq l_S$ and $(l_S \circ \tilde{\mathfrak{S}}) \subseteq l_S$, l_S is an S-int bi-ideal $\Leftrightarrow (l_S \circ l_S) \subseteq l_S$ and $(l_S \circ \tilde{\mathfrak{S}} \circ l_S) \subseteq l_S$,

- 1) l_S is an S-int interior ideal $\Leftrightarrow (\tilde{\mathfrak{S}} \circ l_S \circ \tilde{\mathfrak{S}}) \subseteq l_S$,

Theorem 2.19 [40, 41]. Let S be a semigroup. Then,

- 2) Every S-int L- (R- /two-sided) ideal is an S-int subsemigroup (S-int bi-ideal/S-int quasi-ideal).
- 3) Every S-int quasi-ideal is an S-int subsemigroup (S-int bi-ideal).

Proposition 2.20 [40]. Let $l_S \in S_S(U)$, α be a subset of U , $Im(l_S)$ be the image of l_S such that $\alpha \in Im(l_S)$. If l_S is an S-int subsemigroup of S , then $\mathcal{U}(l_S; \alpha)$ is a subsemigroup of S .

3. SOFT INTERSECTION QUASI-INTERIOR IDEALS OF SEMIGROUPS

In this section, we presented the concept of soft intersection quasi-interior ideals of semigroups, provided its examples, thoroughly examined its relationships with other soft intersection ideals, and analyzed them in terms of certain SS concepts and operations.

Definition 3.1. A soft set f_S over U is called a soft intersection left quasi-interior ideal of S over U if $f_S(x\eta z) \supseteq f_S(\eta) \cap f_S(z)$, and is called a soft intersection right quasi-interior ideal of S over U if $f_S(x\eta z) \supseteq f_S(x) \cap f_S(z)$ for all $x, \eta, z, \eta \in S$. A soft set over U is called a soft intersection quasi-interior ideal of S if it is both soft intersection left-quasi-interior ideal and soft intersection right quasi-interior ideal of S over U .

For the sake of brevity, soft intersection left (right) quasi-interior ideal of S over U is abbreviated by S-int L-Q| (R-Q|), and soft intersection quasi-interior ideal is abbreviated by S-int Q| ideal.

Example 3.2. Consider the semigroup $S = \{b, \tilde{\omega}, \varsigma\}$ defined by the following table:

\cdot	b	$\tilde{\omega}$	ς
b	b	b	b
$\tilde{\omega}$	b	$\tilde{\omega}$	$\tilde{\omega}$
ς	b	$\tilde{\omega}$	$\tilde{\omega}$

Let f_S, l_S and g_S be SSs over $U = \mathbb{Z}$ as follows:

$$f_S = \{(b, \{1,5,7,11\}), (\tilde{\omega}, \{1,5,7\}), (\varsigma, \{1,5\})\}, \quad l_S = \{(b, \{3,8,9\}), (\tilde{\omega}, \{3,9\}), (\varsigma, \{9\})\} \text{ and } g_S = \{(b, \{6,12\}), (\tilde{\omega}, \{2,4,6,10\}), (\varsigma, \{13\})\}.$$

Then, one can easily show that f_S and l_S is an S-int Q| ideal of S . Here, we find it appropriate to give a few concrete examples of elements for ease of illustration

in order to be more understandable. In fact, $f_S(b\tilde{\omega}\varsigma b) = f_S(b) \supseteq f_S(\tilde{\omega}) \cap f_S(b)$, $f_S(\tilde{\omega}\varsigma\tilde{\omega}b) = f_S(b) \supseteq f_S(\varsigma) \cap f_S(b)$ and $f_S(\varsigma\varsigma\varsigma\varsigma) = f_S(\tilde{\omega}) \supseteq f_S(\varsigma) \cap f_S(\varsigma)$.

It can be easily shown that the SS f_S satisfies the S-int L-Q| ideal condition for all other element combinations of the set S . Similarly, $f_S(\varsigma\tilde{\omega}b) = f_S(b) \supseteq f_S(\varsigma) \cap f_S(\tilde{\omega})$, $f_S(b\tilde{\omega}b\tilde{\omega}) = f_S(b) \supseteq f_S(b) \cap f_S(b)$ and $f_S(\tilde{\omega}\tilde{\omega}\varsigma b) = f_S(b) \supseteq f_S(\tilde{\omega}) \cap f_S(\varsigma)$.

It can be easily shown that the SS f_S satisfies the S-int R-Q| ideal condition for all other element combinations of the set S , thus f_S is an S-int Q| ideal.

Similarly, $l_S(\varsigma b\varsigma b) = l_S(b) \supseteq l_S(b) \cap l_S(b)$, $l_S(b\tilde{\omega}\tilde{\omega}\tilde{\omega}) = l_S(b) \supseteq l_S(b) \cap l_S(\tilde{\omega})$ and $l_S(\varsigma\tilde{\omega}\tilde{\omega}\tilde{\omega}) = l_S(\tilde{\omega}) \supseteq l_S(\tilde{\omega}) \cap l_S(\tilde{\omega})$.

It can be easily shown that the SS l_S satisfies the S-int L-Q| ideal condition for all other element combinations of the set S . Similarly, $l_S(\tilde{\omega}\tilde{\omega}\tilde{\omega}\tilde{\omega}) = l_S(\tilde{\omega}) \supseteq l_S(\tilde{\omega}) \cap l_S(\tilde{\omega})$, $l_S(\tilde{\omega}b\tilde{\omega}\tilde{\omega}) = l_S(b) \supseteq l_S(\tilde{\omega}) \cap l_S(b)$ and $l_S(b\tilde{\omega}\tilde{\omega}b) = l_S(b) \supseteq l_S(b) \cap l_S(\tilde{\omega})$.

It can be easily shown that the SS l_S satisfies the S-int R-Q| ideal condition for all other element combinations of the set S , thus l_S is an S-int Q| ideal. However, since $g_S(b\tilde{\omega}b\tilde{\omega}) = g_S(b) \not\supseteq g_S(\tilde{\omega}) \cap g_S(\tilde{\omega})$. g_S is not an S-int Q| ideal.

It is well known that a subsemigroup Z of a semigroup S is an L- (R-) Q| ideal of S if $SZS \subseteq Z$ ($ZSZ \subseteq Z$). It is natural to extend this property to semigroup theory with Proposition 3.3, Proposition 3.4 and Theorem 3.5.

Proposition 3.3. Let $f_S \in S_S(U)$. Then, f_S is an S-int L-Q| ideal iff $\tilde{S} \circ_{f_S} \tilde{S} \circ_{f_S} \tilde{S} \subseteq f_S$.

Proof: Suppose that f_S is an S-int L-Q| ideal and $a \in S$. If $(\tilde{S} \circ_{f_S} \tilde{S} \circ_{f_S})(a) = \emptyset$, then $\tilde{S} \circ_{f_S} \tilde{S} \circ_{f_S} \subseteq f_S$. Otherwise, there exist elements $x, y, h, n, r, s \in S$ such that $a = xy$, $x = hn$ and $n = rs$, for $a \in S$. Since f_S is an S-int L-Q| ideal, $f_S(a) = f_S(xy) = f_S((hn)y) = f_S(h(rs)y) \supseteq f_S(r) \cap f_S(y)$. Therefore,

$$\begin{aligned} (\tilde{S} \circ_{f_S} \tilde{S} \circ_{f_S})(a) &= [(\tilde{S} \circ_{f_S} \tilde{S}) \circ_{f_S}](a) \\ &= \bigcup_{a=xy} \{(\tilde{S} \circ_{f_S} \tilde{S})(x) \cap f_S(y)\} \\ &= \bigcup_{a=xy} \left\{ \bigcup_{x=hn} \{ \tilde{S}(h) \cap (f_S \circ \tilde{S})(n) \} \right. \\ &\quad \left. \cap f_S(y) \right\} \end{aligned}$$

$$\begin{aligned}
 &= \bigcup_{a=xy} \left\{ \bigcup_{x=hn} \left\{ \tilde{\mathfrak{S}}(h) \right. \right. \\
 &\quad \left. \left. \cap \bigcup_{n=rs} \left\{ \mathfrak{F}_S(r) \cap \tilde{\mathfrak{S}}(s) \right\} \right\} \cap \mathfrak{F}_S(y) \right\} \\
 &= \bigcup_{a=hrsy} \left\{ \mathfrak{F}_S(r) \cap \mathfrak{F}_S(y) \right\} \\
 &\subseteq \bigcup_{a=hrsy} \left\{ \mathfrak{F}_S(hrsy) \right\} \\
 &= \mathfrak{F}_S(xy) \\
 &= \mathfrak{F}_S(a)
 \end{aligned}$$

Thus, we have $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$. Moreover, in the case where $a = xy$ and $x \neq hn$ for $a \in S$, since $(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(x) = \emptyset$, $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$ is satisfied.

Conversely, assume that $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$. Let $a = xyzt$ for $a, x, y, z, t \in S$. Then, we have

$$\begin{aligned}
 \mathfrak{F}_S(xyzt) &= \mathfrak{F}_S(a) \\
 &\supseteq (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(a) \\
 &= [(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}}) \circ_{\mathfrak{F}_S}](a) \\
 &= \bigcup_{a=hn} \left\{ (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(h) \cap \mathfrak{F}_S(n) \right\} \\
 &\supseteq (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(xyz) \cap \mathfrak{F}_S(t) \\
 &= \bigcup_{xyz=bc} \left\{ (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(b) \cap \tilde{\mathfrak{S}}(c) \right\} \cap \mathfrak{F}_S(t) \\
 &\supseteq [(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(xy) \cap \tilde{\mathfrak{S}}(z)] \cap \mathfrak{F}_S(t) \\
 &= \left[\bigcup_{xy=pq} \left\{ \tilde{\mathfrak{S}}(p) \cap \mathfrak{F}_S(q) \right\} \cap \tilde{\mathfrak{S}}(z) \right] \cap \mathfrak{F}_S(t) \\
 &\supseteq \left\{ \tilde{\mathfrak{S}}(x) \cap \mathfrak{F}_S(y) \right\} \cap \tilde{\mathfrak{S}}(z) \cap \mathfrak{F}_S(t) \\
 &= \{U \cap \mathfrak{F}_S(y)\} \cap U \cap \mathfrak{F}_S(t) \\
 &= \mathfrak{F}_S(y) \cap \mathfrak{F}_S(t)
 \end{aligned}$$

Hence, $\mathfrak{F}_S(xyzt) \supseteq \mathfrak{F}_S(y) \cap \mathfrak{F}_S(t)$ implying that \mathfrak{F}_S is an S-int L-Q| ideal.

Proposition 3.4. Let $\mathfrak{F}_S \in \mathcal{S}_S(U)$. Then, \mathfrak{F}_S is an S-int R-Q| ideal iff $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$.

Proof: Assume that \mathfrak{F}_S is an S-int R-Q| ideal and $a \in S$. If $(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(a) = \emptyset$, then $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$. Otherwise, there exist elements $x, y, h, n, r, s \in S$ such that $a = xy$, $y = hn$ and $n = rs$, for $a \in S$. Since \mathfrak{F}_S is an S-int R-Q| ideal of S , $\mathfrak{F}_S(a) = \mathfrak{F}_S(xy) = \mathfrak{F}_S(x(hn)) = \mathfrak{F}_S(xh(rs)) \supseteq \mathfrak{F}_S(x) \cap \mathfrak{F}_S(r)$. Thus,

$$\begin{aligned}
 (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(a) &= [\mathfrak{F}_S \circ (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})](a) \\
 &= \bigcup_{a=xy} \left\{ \mathfrak{F}_S(x) \cap (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(y) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \bigcup_{a=xy} \left\{ \mathfrak{F}_S(x) \right. \\
 &\quad \left. \cap \bigcup_{y=hn} \left\{ \tilde{\mathfrak{S}}(h) \cap (\mathfrak{F}_S \circ \tilde{\mathfrak{S}})(n) \right\} \right\} \\
 &= \bigcup_{a=xy} \left\{ \mathfrak{F}_S(x) \right. \\
 &\quad \left. \cap \bigcup_{y=hn} \left\{ \tilde{\mathfrak{S}}(h) \cap \bigcup_{n=rs} \left\{ \mathfrak{F}_S(r) \cap \tilde{\mathfrak{S}}(s) \right\} \right\} \right\} \\
 &= \bigcup_{a=xhrs} \left\{ \mathfrak{F}_S(x) \cap \mathfrak{F}_S(r) \right\} \\
 &\subseteq \bigcup_{a=xhrs} \left\{ \mathfrak{F}_S(xhrs) \right\} \\
 &= \mathfrak{F}_S(xy) \\
 &= \mathfrak{F}_S(a)
 \end{aligned}$$

Hence, we have $\mathfrak{F}_S \circ \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \subseteq \mathfrak{F}_S$. Moreover, in the case where $a = xy$ and $y \neq hn$ for $a \in S$, since $(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(y) = \emptyset$, $\mathfrak{F}_S \circ \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \subseteq \mathfrak{F}_S$ is satisfied.

Conversely, let $\mathfrak{F}_S \circ \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \subseteq \mathfrak{F}_S$. Let $a = xyzt$ for $a, x, y, z, t \in S$. Then, we have

$$\begin{aligned}
 \mathfrak{F}_S(xyzt) &= \mathfrak{F}_S(a) \\
 &\supseteq (\mathfrak{F}_S \circ \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(a) \\
 &= [\mathfrak{F}_S \circ (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})](a) \\
 &= \bigcup_{a=hn} \left\{ \mathfrak{F}_S(h) \cap (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(n) \right\} \\
 &\supseteq \mathfrak{F}_S(x) \cap (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}})(yzt) \\
 &= \mathfrak{F}_S(x) \cap \bigcup_{yzt=de} \left\{ (\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(d) \cap \tilde{\mathfrak{S}}(e) \right\} \\
 &\supseteq \mathfrak{F}_S(x) \cap [(\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S})(yz) \cap \tilde{\mathfrak{S}}(t)] \\
 &= \mathfrak{F}_S(x) \cap \left[\bigcup_{yz=pq} \left\{ \tilde{\mathfrak{S}}(p) \cap \mathfrak{F}_S(q) \right\} \cap \tilde{\mathfrak{S}}(t) \right] \\
 &\supseteq \mathfrak{F}_S(x) \cap \left\{ \tilde{\mathfrak{S}}(y) \cap \mathfrak{F}_S(z) \right\} \cap \tilde{\mathfrak{S}}(t) \\
 &= \mathfrak{F}_S(x) \cap \{U \cap \mathfrak{F}_S(z)\} \cap U \\
 &= \mathfrak{F}_S(x) \cap \mathfrak{F}_S(z)
 \end{aligned}$$

Therefore, $\mathfrak{F}_S(xyzt) \supseteq \mathfrak{F}_S(x) \cap \mathfrak{F}_S(z)$, implying that \mathfrak{F}_S is an S-int R-Q| ideal.

Theorem 3.5. Let $\mathfrak{F}_S \in \mathcal{S}_S(U)$. Then, \mathfrak{F}_S is an S-int Q| ideal iff $\tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \subseteq \mathfrak{F}_S$ and $\mathfrak{F}_S \circ \tilde{\mathfrak{S}} \circ_{\mathfrak{F}_S} \tilde{\mathfrak{S}} \subseteq \mathfrak{F}_S$.

Proof: It follows from Proposition 3.3 and Proposition 3.4.

Corollary 3.6. $\tilde{\mathfrak{S}}$ and \emptyset_S are S-int Q| ideals.

Theorem 3.7. Let \mathcal{Z} be a subsemigroup of S . Then, \mathcal{Z} is a Q| ideal of S iff $S_{\mathcal{Z}}$, the \square CF of \mathcal{Z} , is an S-int Q| ideal.

Proof: Let \mathcal{Z} be a Q| ideal of S . Then, $S\mathcal{Z}S\mathcal{Z} \subseteq \mathcal{Z}$ and $\mathcal{Z}S\mathcal{Z}S \subseteq \mathcal{Z}$. By Theorem 2.12, $\tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}} = S_S \circ S_{\mathcal{Z}} \circ S_S \circ S_{\mathcal{Z}} = S_{S\mathcal{Z}S\mathcal{Z}} \subseteq S_{\mathcal{Z}}$ and $S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S} = S_{\mathcal{Z}} \circ S_S \circ S_{\mathcal{Z}} \circ S_S = S_{\mathcal{Z}S\mathcal{Z}S} \subseteq S_{\mathcal{Z}}$. Hence, $S_{\mathcal{Z}}$ is an S-int Q| ideal.

Conversely, let $S_{\mathcal{Z}}$ be an S-int Q| ideal and \mathcal{Z} be a subsemigroup of S . Then, $\tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}} \subseteq S_{\mathcal{Z}}$ and $S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S} \subseteq S_{\mathcal{Z}}$. Let $x \in S\mathcal{Z}S\mathcal{Z}$. Then, $S_{\mathcal{Z}}(x) \supseteq (\tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}})(x) = (S_S \circ S_{\mathcal{Z}} \circ S_S \circ S_{\mathcal{Z}})(x) = S_{S\mathcal{Z}S\mathcal{Z}}(x) = U$. Thus, $S_{\mathcal{Z}}(x) = U$ and so $x \in \mathcal{Z}$, implying that $S\mathcal{Z}S\mathcal{Z} \subseteq \mathcal{Z}$. Hence, \mathcal{Z} is an L-Q| ideal of S . Similarly, let $y \in \mathcal{Z}S\mathcal{Z}S$. Then, $S_{\mathcal{Z}}(y) \supseteq (S_{\mathcal{Z}} \circ \tilde{S} \circ S_{\mathcal{Z}} \circ \tilde{S})(y) = (S_{\mathcal{Z}} \circ S_S \circ S_{\mathcal{Z}} \circ S_S)(y) = S_{\mathcal{Z}S\mathcal{Z}S}(y) = U$. Thus, $S_{\mathcal{Z}}(y) = U$, and so $y \in \mathcal{Z}$, implying that $\mathcal{Z}S\mathcal{Z}S \subseteq \mathcal{Z}$. Hence, \mathcal{Z} is an R-Q| ideal of S . Therefore, \mathcal{Z} is a Q| ideal of S .

Example 3.8. We consider the semigroup in Example 3.2. One can show that $A = \{b, \tilde{\omega}\}$ is a Q| ideal of S . By the definition of \square CF, $S_A = \{(b, U), (\tilde{\omega}, U), (\zeta, \emptyset)\}$. One can easily show that S_A is an S-int Q| ideal. Conversely, by choosing the S-int Q| ideal as $f_S = \{(b, U), (\tilde{\omega}, \emptyset), (\zeta, \emptyset)\}$, which is the \square CF of $K = \{b\}$, one can show that K is a Q| ideal of S .

Now, we continue with the relationships between S-int Q| ideals and other types of S-int ideals of S .

Proposition 3.9. Every S-int interior ideal is an S-int L-Q| ideal.

Proof: Let f_S be an S-int interior ideal. Then, $\tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$. Since, $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq \tilde{S} \circ f_S \circ \tilde{S} \subseteq \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$. Hence, f_S is an S-int L-Q| ideal of S .

We show with a counterexample that the converse of Proposition 3.9 is not true:

Example 3.10. Consider the semigroup $S = \{\mathcal{R}, \mathcal{I}, r, \check{v}\}$ defined by the following table

.	\mathcal{R}	\mathcal{I}	r	\check{v}
\mathcal{R}	\mathcal{R}	\mathcal{R}	\mathcal{R}	\mathcal{R}
\mathcal{I}	\mathcal{R}	\mathcal{R}	\mathcal{R}	\mathcal{R}
r	\mathcal{R}	\mathcal{R}	\mathcal{R}	\mathcal{I}
\check{v}	\mathcal{R}	\mathcal{R}	\mathcal{I}	r

Let f_S be a SS over $U = \mathbb{Z}$ as follows:

$$f_S = \left\{ (\mathcal{R}, \{-5, -4, -3, -1\}), (\mathcal{I}, \{-4, -3\}), (r, \{-5\}), (\check{v}, \{-5, -1\}) \right\}$$

Here, f_S is an S-int L-Q| ideal. In fact,

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(\mathcal{R}) = \{-5, -4, -3, -1\} \subseteq f_S(\mathcal{R}) = \{-5, -4, -3, -1\}$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(\mathcal{I}) = \emptyset \subseteq f_S(\mathcal{I}) = \{-4, -3\}$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(r) = \emptyset \subseteq f_S(r) = \{-5\}$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(\check{v}) = \emptyset \subseteq f_S(\check{v}) = \{-5, -1\}$$

thus, f_S is an S-int L-Q| ideal of S . However, since $f_S(\check{v}\check{v}\check{v}) = f_S(\mathcal{I}) \not\subseteq f_S(\check{v})$. f_S is not an S-int interior ideal.

Proposition 3.11 shows that the converse of Proposition 3.9 holds for L-simple semigroups.

Proposition 3.11. Let $f_S \in S_S(U)$ and S be an L-simple semigroup. Then, the following conditions are equivalent:

- f_S is an S-int interior ideal.
- f_S is an S-int L-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.9. Assume that f_S is an S-int L-Q| ideal. Let $a, b, c, d \in S$. By assumption, for every $b, c \in S$, there exists $x \in S$ such that $c = xb$. Since S is an L-simple semigroup, $f_S(abc) = f_S(ab(xb)) \supseteq f_S(b) \cap f_S(b) = f_S(b)$. Hence, f_S is an S-int interior ideal.

Proposition 3.12. Every S-int interior ideal is an S-int R-Q| ideal.

Proof: Let f_S be an S-int interior ideal. Then, $\tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$. Since, $f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \subseteq \tilde{S} \circ f_S \circ \tilde{S} \subseteq \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$. Hence, f_S is an S-int R-Q| ideal of S .

We show with a counterexample that the converse of Proposition 3.12 is not true:

Example 3.13. Consider the SS f_S in Example 3.10. The SS f_S in Example 3.10 is an S-int R-Q| ideal. In fact,

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(\mathcal{R}) = \{-5, -4, -3, -1\} \subseteq f_S(\mathcal{R}) = \{-5, -4, -3, -1\}$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(\mathcal{I}) = \emptyset \subseteq f_S(\mathcal{I}) = \{-4, -3\}$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(r) = \emptyset \subseteq f_S(r) = \{-5\}$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(\check{v}) = \emptyset \subseteq f_S(\check{v}) = \{-5, -1\}$$

thus, f_S is an S-int R-Q| ideal of S . However, since $f_S(\check{v}\check{v}\check{v}) = f_S(\mathcal{I}) \not\subseteq f_S(\check{v})$. f_S is not an S-int interior ideal.

Proposition 3.14 shows that the converse of Proposition 3.12 holds for R-simple semigroups.

Proposition 3.14. Let $f_S \in S_S(U)$ and S be an R-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int interior ideal.
2. f_S is an S-int R-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.12. Assume that f_S is an S-int R-Q| ideal and $a, b, c, d \in S$. By assumption, for every $a, b \in S$, there exists $y \in S$ such that $a = by$. Thus, $f_S(abc) = f_S((by)bc) \supseteq f_S(b) \cap f_S(b) = f_S(b)$. Hence, f_S is an S-int interior ideal.

Theorem 3.15. Every S-int interior ideal is an S-int Q| ideal.

Proof: It follows from Proposition 3.9 and Proposition 3.12.

Here note that the converse of Theorem 3.15 is not true follows from Example 3.10 and Example 3.13. Theorem 3.16 shows that the converse of Theorem 3.15 holds for groups as well.

Theorem 3.16. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-int interior ideal.
2. f_S is an S-int Q| ideal.

Proof: (1) implies (2) is obvious by Theorem 3.15. Assume that f_S is an S-int Q| ideal of a group S . Then, by Theorem 2.1 (2), S is both an L-simple and an R-simple semigroup. The rest of the proof follows from Proposition 3.11 and Proposition 3.14.

Proposition 3.17. Every S-int L-ideal is an S-int L-Q| ideal.

Proof: Let f_S be an S-int L-ideal of S . Then, $\tilde{S} \circ f_S \subseteq f_S$ ve $f_S \circ f_S \subseteq f_S$. Thus, $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq f_S \circ f_S \subseteq f_S$. Hence, f_S is an S-int L-Q| ideal of S .

We show with a counterexample that the converse of Proposition 3.17 is not true:

Example 3.18. Consider the semigroup $S = \{z, h, r, a\}$ defined by the following table:

*	z	h	r	A
z	z	z	z	z
h	z	z	z	z
r	z	z	h	z
a	z	z	h	h

Let f_S be a SS over $U = \mathbb{Z}_8^*$ as follows:

$$f_S = \{(z, \mathbb{Z}_8^*), (h, \{\bar{5}\}), (r, \{\bar{1}, \bar{3}\}), (a, \{\bar{1}, \bar{7}\})\}$$

Here, f_S is an S-int L-Q| ideal. In fact,

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(z) = \mathbb{Z}_8^* \subseteq f_S(z) = \mathbb{Z}_8^*$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(h) = \emptyset \subseteq f_S(h) = \{\bar{5}\}$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(r) = \emptyset \subseteq f_S(r) = \{\bar{1}, \bar{3}\}$$

$$(\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(a) = \emptyset \subseteq f_S(a) = \{\bar{1}, \bar{7}\}$$

thus, f_S is an S-int L-Q| ideal of S . However, since $f_S(ar) = f_S(h) \not\subseteq f_S(r)$. f_S is not an S-int L-ideal.

Proposition 3.19 shows that the converse of Proposition 3.17 holds for the idempotent SSs, and Proposition 3.20 shows that the converse of Proposition 3.17 holds for the regular semigroups as well.

Proposition 3.19. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-int L-ideal.
2. f_S is an S-int L-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.17. Let f_S be an S-int L-Q|ideal of S . Since f_S is an idempotent S-int L-Q| ideal, $\tilde{S} \circ f_S = \tilde{S} \circ f_S \circ f_S = \tilde{S} \circ f_S \circ f_S \circ f_S \subseteq \tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq f_S$. Hence, f_S is an S-int L-ideal.

Proposition 3.20. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int L-ideal.
2. f_S is an S-int L-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.17. Assume that f_S is an S-int L-Q| ideal and $a, b \in S$. By assumption, for every $b \in S$, there exists $x \in S$ such that $b = bxb$. Thus, $f_S(ab) = f_S(a(bxb)) \supseteq f_S(b) \cap f_S(b) = f_S(b)$. Thus, f_S is an S-int L-ideal.

Proposition 3.21. Every S-int R-ideal is an S-int R-Q| ideal.

Proof: Let f_S be an S-int R-ideal of S . Then, $f_S \circ \tilde{S} \subseteq f_S$ ve $f_S \circ f_S \subseteq f_S$. Thus, $f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S \circ f_S \subseteq f_S$. Hence, f_S is an S-int R-Q| ideal of S .

We show with a counterexample that the converse of Proposition 3.21 is not true:

Example 3.22. Consider the SS f_S in Example 3.18. The SS f_S in Example 3.18. is an S-int R-Q| ideal. In fact,

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(z) = \mathbb{Z}_8^* \subseteq f_S(z) = \mathbb{Z}_8^*$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(b) = \emptyset \subseteq f_S(b) = \{\bar{5}\}$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(r) = \emptyset \subseteq f_S(r) = \{\bar{1}, \bar{3}\}$$

$$(f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(a) = \emptyset \subseteq f_S(a) = \{\bar{1}, \bar{7}\}$$

thus, f_S is an S-int R-Q| ideal of S . However, since $f_S(ax) = f_S(b) \not\subseteq f_S(a)$. f_S is not an S-int R-ideal.

Proposition 3.23 shows that the converse of Proposition 3.21 holds for the idempotent SSs, and Proposition 3.24 shows that the converse of Proposition 3.21 holds for the regular semigroups as well.

Proposition 3.23. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-int R-ideal.
2. f_S is an S-int R-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.21. Let f_S be an S-int R-Q| ideal of S . Since f_S is an idempotent S-int R-Q| ideal, $f_S \circ \tilde{S} = f_S \circ f_S \circ \tilde{S} = f_S \circ f_S \circ \tilde{S} \subseteq f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$. Hence, f_S is an S-int R-ideal.

Proposition 3.24. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int R-ideal.
2. f_S is an S-int R-Q| ideal.

Proof: (1) implies (2) is obvious by Proposition 3.21. Assume that f_S is an S-int R-Q| ideal and $a, b \in S$. By assumption, for every $a \in S$, there exists $x \in S$ such that $a = axa$. Thus, $f_S(ab) = f_S((axa)b) \supseteq f_S(a) \cap f_S(a) = f_S(a)$. Thus, f_S is an S-int R-ideal.

Theorem 3.25. Every S-int ideal is an S-int Q| ideal.

Proof: It follows from Proposition 3.17 and Proposition 3.21.

Here note that the converse of Theorem 3.25 is not true follows from Example 3.18 and Example 3.22. Theorem 3.26 shows that the converse of Theorem 3.25 holds for the idempotent SSs, and Theorem 3.27 shows that the converse of Theorem 3.25 holds for the regular semigroups as well.

Theorem 3.26. Let f_S be an idempotent SS over U . Then, the following conditions are equivalent:

1. f_S is an S-int ideal.
2. f_S is an S-int Q| ideal.

Proof: (1) implies (2) is obvious by Theorem 3.25. (2) implies (1) is obvious by Proposition 3.19 and Proposition 3.23.

Theorem 3.27. Let $f_S \in S_S(U)$ and S be a regular semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int ideal.
2. f_S is an S-int Q| ideal.

Proof: (1) implies (2) is obvious by Theorem 3.25. (2) implies (1) is obvious by Proposition 3.20 and Proposition 3.24.

Proposition 3.28. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-int bi-ideal.
2. f_S is an S-int L-Q| ideal.

Proof: Assume that f_S is an S-int bi-ideal and $a, b, c, d \in S$. By assumption, for every $a, b, c \in S$, there exists $x, y \in S$ such that $a = bx$ and $c = by$. Thus, $f_S(abcd) = f_S((bx)b(by)d) \supseteq f_S(bxb) \cap f_S(byd) \supseteq (f_S(b) \cap f_S(b)) \cap (f_S(b) \cap f_S(d)) = f_S(b) \cap f_S(d)$. Thus, f_S is an S-int L-Q| ideal.

Conversely, assume that f_S is an S-int L-Q| ideal and $a, b \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $a = xb$ and $b = xa$. Then, $f_S(ab) = f_S((xb)(xa)) = f_S(xbxa) \supseteq f_S(b) \cap f_S(a)$.

Thus, f_S is an S-int subsemigroup. Moreover, f_S is an S-int L-Q| ideal and $a, b, c \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $b = ax$. Thus, $f_S(abc) = f_S(a(ax)c) = f_S(aaxc) \supseteq f_S(a) \cap f_S(c)$. Thus, f_S is an S-int bi-ideal.

Proposition 3.29. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-int bi-ideal.
2. f_S is an S-int R-Q| ideal.

Proof: Assume that f_S is an S-int bi-ideal and $a, b, c, d \in S$. By assumption, for every $b, c, d \in S$, there exists $x, y \in S$ such that $b = xc$ and $d = yc$. Thus, $f_S(abcd) = f_S(a(xc)c(yc)) \supseteq f_S(axc) \cap f_S(cyc) \supseteq (f_S(a) \cap f_S(c)) \cap (f_S(c) \cap f_S(c)) = f_S(a) \cap f_S(c)$. Thus, f_S is an S-int R-Q| ideal.

Conversely, assume that f_S is an S-int R-Q| ideal and $a, b \in S$. By assumption, for every $a, b \in S$, there exists $x \in S$ such that $a = bx$ and $b = ax$. Then, $f_S(ab) = f_S((bx)(ax)) = f_S(bxax) \supseteq f_S(b) \cap f_S(a)$. Thus, f_S is an S-int subsemigroup.

Moreover, f_S is an S-int R-Q| ideal and $a, b, c \in S$. By assumption, for every $b, c \in S$, there exists $x \in S$ such that $b = xc$. Thus, $f_S(abc) = f_S(a(xc)c) = f_S(axcc) \supseteq f_S(a) \cap f_S(c)$. Thus, f_S is an S-int bi-ideal.

Theorem 3.30. Let $f_S \in S_S(U)$ and S be a group. Then, the following conditions are equivalent:

1. f_S is an S-int bi-ideal.
2. f_S is an S-int Q| ideal.

Proof: It is implied from the proof of Proposition 3.28 and Proposition 3.29.

Proposition 3.31. Let f_S be an idempotent SS over U . Every S-int L-Q| ideal is an S-int bi-ideal.

Proof: f_S is an idempotent S-int L-Q| ideal, $f_S \circ f_S = f_S \tilde{\subseteq} f_S$. Hence, f_S is an S-int subsemigroup. Moreover, $f_S \circ \tilde{S} \circ f_S = f_S \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} \tilde{S} \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} f_S$. Hence, f_S is an S-int bi-ideal.

Proposition 3.32. Let f_S be an idempotent SS over U . Every S-int R-Q| ideal is an S-int bi-ideal.

Proof: Let f_S be an idempotent S-int R-Q| ideal, $f_S \circ f_S = f_S \tilde{\subseteq} f_S$. Hence, f_S is an S-int subsemigroup. Moreover, $f_S \circ \tilde{S} \circ f_S = f_S \circ \tilde{S} \circ f_S \circ f_S \tilde{\subseteq} f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \tilde{\subseteq} f_S$. Hence, f_S is an S-int bi-ideal.

Theorem 3.33. Let f_S be an idempotent SS over U . Every S-int Q| ideal is an S-int bi-ideal.

Proof: It is implied from the proof of Proposition 3.31 and Proposition 3.32.

Proposition 3.34. Let $f_S \in S_S(U)$ and S be a regular R-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int quasi ideal.
2. f_S is an S-int L-Q| ideal.

Proof: Let f_S be an S-int quasi-ideal. Then, by Theorem 2.19 (2), f_S is an S-int bi-ideal. Since, S is an R-simple semigroup, by Proposition 3.28, f_S is an S-int L-Q| ideal.

Conversely, let f_S be an S-int L-Q| ideal. Since, S is a regular semigroup, by Proposition 3.20, f_S is an S-int L-ideal. Then, by Theorem 2.19 (1), f_S is an S-int quasi ideal.

Proposition 3.35. Let $f_S \in S_S(U)$ and S be a regular L-simple semigroup. Then, the following conditions are equivalent:

1. f_S is an S-int quasi ideal.
2. f_S is an S-int R-Q| ideal.

Proof: Let f_S be an S-int quasi ideal. Then, by Theorem 2.19 (2), f_S is an S-int bi-ideal. Since, S is an L-simple semigroup, by Proposition 3.29, f_S is an S-int R-Q| ideal.

Conversely, let f_S be an S-int R-Q| ideal. Since, S is a regular semigroup, by Proposition 3.24, f_S is an S-int R-ideal. Then, by Theorem 2.19 (1), f_S is an S-int quasi ideal.

Theorem 3.36. Let $f_S \in S_S(U)$ and S be a regular group. Then, the following conditions are equivalent:

1. f_S is an S-int quasi ideal.
2. f_S is an S-int Q| ideal.

Proof: By Theorem 2.1 (2), S is both an R-simple and an L-simple semigroup. The rest of the proof follows from Proposition 3.34 and Proposition 3.35.

Proposition 3.37. Let f_S be an idempotent SS over U . Every S-int L-Q| ideal is an S-int L-bi-quasi ideal.

Proof: Let f_S be an idempotent S-int L-Q| ideal, $(\tilde{S} \circ f_S) \tilde{\cap} (f_S \circ \tilde{S} \circ f_S) \tilde{\subseteq} f_S \circ \tilde{S} \circ f_S = f_S \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} \tilde{S} \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} f_S$. Hence, f_S is an S-int L-bi-quasi ideal.

Proposition 3.38. Let f_S be an idempotent SS over U . Every S-int R-Q| ideal is an S-int R-bi-quasi ideal.

Proof: Let f_S be an idempotent S-int R-Q| ideal, $(f_S \circ \tilde{S}) \tilde{\cap} (f_S \circ \tilde{S} \circ f_S) \tilde{\subseteq} f_S \circ \tilde{S} \circ f_S = f_S \circ \tilde{S} \circ f_S \circ f_S \tilde{\subseteq} f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \tilde{\subseteq} f_S$. Hence, f_S is an S-int R-bi-quasi ideal.

Theorem 3.39. Let f_S be an idempotent SS over U . Every S-int Q| ideal is an S-int bi-quasi ideal.

Proof: It is implied from the proof of Proposition 3.37 and Proposition 3.38.

Proposition 3.40. Let f_S be an idempotent SS over U . Every S-int L-Q| ideal is an S-int bi-interior ideal.

Proof: Let f_S be an idempotent S-int L-Q| ideal, $(\tilde{S} \circ f_S \circ \tilde{S}) \tilde{\cap} (f_S \circ \tilde{S} \circ f_S) \tilde{\subseteq} f_S \circ \tilde{S} \circ f_S = f_S \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} \tilde{S} \circ f_S \circ \tilde{S} \circ f_S \tilde{\subseteq} f_S$. Hence, f_S is an S-int bi-interior ideal.

$(\tilde{S} \circ \tilde{S}) \circ f_S \circ g_S \subseteq (\tilde{S} \circ f_S \circ \tilde{S} \circ f_S) \circ g_S \subseteq f_S \circ g_S$. Hence, $f_S \circ g_S$ is an S-int L-Q| ideal.

Proposition 3.54. Let g_S be a SS over U and f_S be an S-int R-Q| ideal. Then, $g_S \circ f_S$ is an S-int R-Q| ideal.

Proof: Let g_S be a SS over U and f_S be an S-int R-Q| ideal. Then, $(g_S \circ f_S) \circ \tilde{S} \circ (g_S \circ f_S) \circ \tilde{S} \subseteq g_S \circ f_S \circ (\tilde{S} \circ \tilde{S}) \circ f_S \circ \tilde{S} \subseteq g_S \circ (f_S \circ \tilde{S} \circ f_S \circ \tilde{S}) \subseteq g_S \circ f_S$. Hence, $g_S \circ f_S$ is an S-int R-Q| ideal.

Theorem 3.55. Let f_S and g_S be S-int Q| ideals. Then, $g_S \circ f_S$ is an S-int Q| ideal.

Proof: Let f_S and g_S be S-int Q| ideals. Then, $\tilde{S} \circ (g_S \circ f_S) \circ \tilde{S} \circ (g_S \circ f_S) \subseteq \tilde{S} \circ g_S \circ (\tilde{S} \circ \tilde{S}) \circ g_S \circ f_S \subseteq (\tilde{S} \circ g_S \circ \tilde{S} \circ g_S) \circ f_S \subseteq g_S \circ f_S$. Hence, $g_S \circ f_S$ is an S-int L-Q| ideal.

Similarly, $(g_S \circ f_S) \circ \tilde{S} \circ (g_S \circ f_S) \circ \tilde{S} \subseteq g_S \circ f_S \circ (\tilde{S} \circ \tilde{S}) \circ g_S \circ f_S \subseteq g_S \circ (f_S \circ \tilde{S} \circ f_S \circ \tilde{S}) \subseteq g_S \circ f_S$. Hence, $g_S \circ f_S$ is an S-int R-Q| ideal. Thus, $g_S \circ f_S$ is an S-int Q| ideal.

Proposition 3.56. Let g_S be a SS over U and f_S be an S-int L-ideal. Then, $f_S \circ g_S$ is an S-int L-Q| ideal.

Proof: Let g_S be a SS over U and f_S be an S-int L-ideal. Then, by Proposition 3.17, f_S is an S-int L-Q| ideal. The rest of the proof is clear from Proposition 3.53.

Proposition 3.57. Let g_S be a SS over U and f_S be an S-int R-ideal. Then, $g_S \circ f_S$ is an S-int R-Q| ideal.

Proof: Let g_S be a SS over U and f_S be an S-int R-ideal. Then, by Proposition 3.21, f_S is an S-int R-Q| ideal. The rest of the proof is clear from Proposition 3.54.

Proposition 3.58. Let g_S be an S-int L-ideal and f_S be an S-int R-ideal. Then, $g_S \circ f_S$ is an S-int Q| ideal.

Proof: Let g_S be an S-int L-ideal and f_S be an S-int R-ideal. Then, by Proposition 3.17, g_S is an S-int L-Q| ideal and by Proposition 3.53, $g_S \circ f_S$ is an S-int L-Q| ideal. Similarly, by Proposition 3.21, f_S is an S-int R-Q| ideal and by Proposition 3.54, $g_S \circ f_S$ is an S-int R-Q| ideal. Thus, $g_S \circ f_S$ is an S-int Q| ideal.

Conclusion 3.59. Let g_S and f_S be S-int ideals. Then, $g_S \circ f_S$ are S-int Q| ideals.

Proposition 3.60. Let f_S be a nonempty SS over U . Then, every soft subset of f_S containing $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S (f_S \circ \tilde{S} \circ f_S \circ \tilde{S})$ is an S-int L- (R-) Q| ideal.

Proof: The proof is presented only for S-int L-Q| ideal, as the proof for S-int R-Q| ideal can be shown similarly. Let f_S be a SS over U and g_S be a soft subset of f_S containing $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S$. Since, $\tilde{S} \circ g_S \circ \tilde{S} \circ g_S \subseteq \tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq g_S$. Hence, g_S is an S-int L-Q| ideal.

Theorem 3.61. Let f_S be a nonempty SS over U . Then, every soft subset of f_S containing $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S$ and $f_S \circ \tilde{S} \circ f_S \circ \tilde{S}$ is an S-int Q| ideal.

Proposition 3.62. Let f_S be an S-int subsemigroup over U , g_S be an S-int L-ideal and $f_S = g_S \circ g_S$. Then, f_S is an S-int L-Q| ideal.

Proof: Let f_S be an S-int subsemigroup over U , g_S be an S-int L-ideal and $f_S = g_S \circ g_S$. Then, $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S = \tilde{S} \circ (g_S \circ g_S) \circ \tilde{S} \circ (g_S \circ g_S) = (\tilde{S} \circ g_S) \circ g_S \circ (\tilde{S} \circ g_S) \circ g_S \subseteq (g_S \circ g_S) \circ (g_S \circ g_S) = f_S \circ f_S \subseteq f_S$. Hence, f_S is an S-int L-Q| ideal.

Proposition 3.63. Let f_S be an S-int subsemigroup over U , g_S be an S-int R-ideal and $f_S = g_S \circ g_S$. Then, f_S is an S-int R-Q| ideal.

Proof: Let f_S be an S-int subsemigroup over U , g_S be an S-int R-ideal and $f_S = g_S \circ g_S$. Then, $f_S \circ \tilde{S} \circ f_S \circ \tilde{S} = (g_S \circ g_S) \circ \tilde{S} \circ (g_S \circ g_S) \circ \tilde{S} = g_S \circ (\tilde{S} \circ g_S) \circ g_S \circ (\tilde{S} \circ g_S) \subseteq (g_S \circ g_S) \circ (g_S \circ g_S) = f_S \circ f_S \subseteq f_S$. Hence, f_S is an S-int R-Q| ideal.

Theorem 3.64. Let f_S be an S-int subsemigroup over U , g_S be an S-int ideal and $f_S = g_S \circ g_S$. Then, f_S is an S-int Q| ideal.

Proposition 3.65. Let f_S be a SS over U and $\tilde{S} \circ \tilde{S} \circ \tilde{S} \circ f_S \subseteq f_S$. Then, f_S is an S-int L-Q| ideal.

Proof: Let f_S be a SS over U and $\tilde{S} \circ \tilde{S} \circ \tilde{S} \circ f_S \subseteq f_S$. Then, $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq \tilde{S} \circ \tilde{S} \circ \tilde{S} \circ f_S \subseteq f_S$. Hence, f_S is an S-int L-Q| ideal.

Proposition 3.66. Let f_S be a SS over U and $f_S \circ \tilde{S} \circ \tilde{S} \circ \tilde{S} \subseteq f_S$. Then, f_S is an S-int R-Q| ideal.

Proof: Let f_S be a SS over U and $f_S \circ \tilde{S} \circ \tilde{S} \circ \tilde{S} \subseteq f_S$. Then, $f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S \circ \tilde{S} \circ \tilde{S} \circ \tilde{S} \subseteq f_S$. Hence, f_S is an S-int R-Q| ideal.

Theorem 3.67. Let f_S be a SS over U , $\tilde{S} \circ \tilde{S} \circ \tilde{S} \circ f_S \subseteq f_S$ and $f_S \circ \tilde{S} \circ \tilde{S} \circ \tilde{S} \subseteq f_S$. Then, f_S is an S-int Q| ideal.

Proposition 3.68. Let f_S be an S-int subsemigroup over U , α be a subset of U , $Im(f_S)$ be the image of f_S

such that $\alpha \in Im(f_S)$. If f_S is an S-int L- (R-) Q| ideal of S , then $\mathcal{U}(f_S; \alpha)$ is an L- (R-) Q| ideal of S .

Proof: The proof is presented only for S-int L-Q| ideal, as the proof for S-int R-Q| ideal can be shown similarly. Since, $f_S(x) = \alpha$ for some $x \in S$, $\emptyset \neq \mathcal{U}(f_S; \alpha) \subseteq S$. Let $k \in [S \cdot \mathcal{U}(f_S; \alpha) \cdot S \cdot \mathcal{U}(f_S; \alpha)]$. Then, there exist $x, y \in \mathcal{U}(f_S; \alpha)$ and $a, b \in S$ such that $k = axby$. Thus, $f_S(x) \supseteq \alpha$, $f_S(y) \supseteq \alpha$. Since f_S is an S-int L-Q| ideal, $f_S(k) = f_S(axby) \supseteq f_S(x) \cap f_S(y) \supseteq \alpha \cap \alpha = \alpha$

Hence, $f_S(k) \supseteq \alpha$, implying that $k \in \mathcal{U}(f_S; \alpha)$. Therefore, $S \cdot \mathcal{U}(f_S; \alpha) \cdot S \cdot \mathcal{U}(f_S; \alpha) \subseteq \mathcal{U}(f_S; \alpha)$. Moreover, since f_S is an S-int subsemigroup over U , by Proposition 2.20, $\mathcal{U}(f_S; \alpha)$ is a subsemigroup of S . Thus, $\mathcal{U}(f_S; \alpha)$ is an L-Q| ideal.

Theorem 3.69. Let f_S be an S-int subsemigroup over U , α be a subset of U , $Im(f_S)$ be the image of f_S such that $\alpha \in Im(f_S)$. If f_S is an S-int Q| ideal of S , then $\mathcal{U}(f_S; \alpha)$ is a Q| ideal of S .

We illustrate Theorem 3.69 with Example 3.70.

Example 3.70. Consider the SS \mathcal{F}_S in Example 3.2. It is clear that The SS \mathcal{F}_S in Example 3.2 an S-int Q| ideal. By considering the image set of \mathcal{F}_S , that is, $Im(\mathcal{F}_S) = \{\{1,5,7,11\}, \{1,5,7\}, \{1,5\}\}$ we obtain the following:

$$\mathcal{U}(\mathcal{F}_S; \alpha) = \begin{cases} \{b, \tilde{\omega}, \zeta\}, & \alpha = \{1,5\} \\ \{b, \tilde{\omega}\}, & \alpha = \{1,5,7\} \\ \{b\}, & \alpha = \{1,5,7,11\} \end{cases}$$

Here, $\{b, \tilde{\omega}, \zeta\}$, $\{b, \tilde{\omega}\}$ and $\{b\}$ are all subsemigroups of S . In fact, since

$$\{b, \tilde{\omega}, \zeta\} \cdot \{b, \tilde{\omega}, \zeta\} = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}, \zeta\},$$

$$\{b, \tilde{\omega}\} \cdot \{b, \tilde{\omega}\} = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}\},$$

$$\{b\} \cdot \{b\} = \{b\} \subseteq \{b\}$$

each $\mathcal{U}(\mathcal{F}_S; \alpha)$ is a subsemigroup of S .

Here, $\{b, \tilde{\omega}, \zeta\}$, $\{b, \tilde{\omega}\}$ and $\{b\}$ are all Q| ideals of S . In fact, since

$$S \cdot \{b, \tilde{\omega}, \zeta\} \cdot S \cdot \{b, \tilde{\omega}, \zeta\} = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}, \zeta\},$$

$$S \cdot \{b, \tilde{\omega}\} \cdot S \cdot \{b, \tilde{\omega}\} = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}\},$$

$$S \cdot \{b\} \cdot S \cdot \{b\} = \{b\} \subseteq \{b\}$$

each $\mathcal{U}(\mathcal{F}_S; \alpha)$ is an L-Q| ideal of S . Similarly, since

$$\{b, \tilde{\omega}, \zeta\} \cdot S \cdot \{b, \tilde{\omega}, \zeta\} \cdot S = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}, \zeta\},$$

$$\{b, \tilde{\omega}\} \cdot S \cdot \{b, \tilde{\omega}\} \cdot S = \{b, \tilde{\omega}\} \subseteq \{b, \tilde{\omega}\},$$

$$\{b\} \cdot S \cdot \{b\} \cdot S = \{b\} \subseteq \{b\}$$

each $\mathcal{U}(\mathcal{F}_S; \alpha)$ is an R-Q| ideal of S , and thus each of $\mathcal{U}(\mathcal{F}_S; \alpha)$ is a Q| ideal of S .

Now, consider the SS g_S in Example 3.2. By taking into account, $Im(g_S) = \{\{6,12\}, \{2,4,6,10\}, \{13\}\}$ we obtain the following:

$$\mathcal{U}(g_S; \alpha) = \begin{cases} \{\zeta\}, & \alpha = \{13\} \\ \{b\}, & \alpha = \{6,12\} \\ \{\tilde{\omega}\}, & \alpha = \{2,4,6,10\} \end{cases}$$

Here, $\{\zeta\}$ is not a Q| ideal of S . In fact, since $S \cdot \{\zeta\} \cdot S \cdot \{\zeta\} = \{b, \tilde{\omega}\} \not\subseteq \{\zeta\}$. One of the $\mathcal{U}(g_S; \alpha)$ is not a L quasi interior ideal of S , hence it is not a Q| ideal of S . It is seen that each of $\mathcal{U}(g_S; \alpha)$ is not a Q| ideal of S . On the other hand, in Example 3.2 it was shown that g_S is not an S-int Q| ideal of S .

Definition 3.71. Let f_S be an S-int L- (R-) Q| ideal of S . Then, the L- (R-) Q| ideals $\mathcal{U}(f_S; \alpha)$ are called upper α -L- (R-) Q| ideals of f_S .

Proposition 3.72. Let f_S be a SS over U , $\mathcal{U}(f_S; \alpha)$ be upper α -L- (R-) Q| ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-int L- (R-) Q| ideal.

Proof: The proof is presented only for S-int L-Q| ideal, as the proof for S-int R-Q| ideal can be shown similarly. Let $a, b, x, y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{U}(f_S; \alpha_1)$ and $y \in \mathcal{U}(f_S; \alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x, y \in \mathcal{U}(f_S; \alpha_1)$ and since $\mathcal{U}(f_S; \alpha)$ is a L-Q| of S for all $\alpha \subseteq U$, it follows that $axby \in \mathcal{U}(f_S; \alpha_1)$. Hence, $f_S(axby) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_S(x) \cap f_S(y)$. Thus, f_S is an S-int L-Q| ideal.

Theorem 3.73. Let f_S be a SS over U , $\mathcal{U}(f_S; \alpha)$ be upper α -Q| ideal of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an S-int Q| ideal.

Proposition 3.74. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_S is an S-int L- (R-) Q| ideal of S , then $\Psi(f_S)$ is an S-int L- (R-) Q| ideal of T .

Proof: The proof is presented only for S-int L-Q| ideal, as the proof for S-int R-Q| ideal can be shown similarly. Let $t_1, t_2, t_3, t_4 \in T$. Since Ψ is surjective, there exist $s_1, s_2, s_3, s_4 \in S$ such that $\Psi(s_1) = t_1$, $\Psi(s_2) = t_2$, $\Psi(s_3) = t_3$ and $\Psi(s_4) = t_4$. Then,

$$\begin{aligned} (\Psi(f_S))(t_1 t_2 t_3 t_4) &= \bigcup \{f_S(s) : s \in S, \Psi(s) = t_1 t_2 t_3 t_4\} \\ &= \bigcup \{f_S(s) : s \in S, s = \Psi^{-1}(t_1 t_2 t_3 t_4)\} \\ &= \bigcup \{f_S(s) : s \in S, s \\ &= \Psi^{-1}(\Psi(s_1 s_2 s_3 s_4)) = s_1 s_2 s_3 s_4\} \end{aligned}$$

$$\begin{aligned}
 &= \bigcup_{\{f_S(s_1 s_2 s_3 s_4): s_i \in S, \Psi(s_i) = t_i, i \\
 &= 1,2,3,4\}} \\
 &\supseteq \bigcup_{\{f_S(s_2) \cap f_S(s_4): s_2, s_4 \in S, \Psi(s_2) \\
 &= t_2 \text{ and } \Psi(s_4) = t_4\}} \\
 &= (\Psi(f_S))(s_2) \cap (\Psi(f_S))(s_4)
 \end{aligned}$$

Hence, $\Psi(f_S)$ is an S-int L-Q| ideal of T .

Theorem 3.75. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_S is an S-int Q| ideal of S , then $\Psi(f_S)$ is an S-int Q| ideal of T .

Proposition 3.76. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_T is an S-int L- (R-) Q| ideal of T , then $\Psi^{-1}(f_T)$ is an S-int L- (R-) Q| ideal of S .

Proof: The proof is presented only for S-int L-Q| ideal, as the proof for S-int R-Q| ideal can be shown similarly. Let $s_1, s_2, s_3, s_4 \in S$. Then,

$$\begin{aligned}
 (\Psi^{-1}(f_T))(s_1 s_2 s_3 s_4) &= f_T(\Psi(s_1 s_2 s_3 s_4)) \\
 &= f_T(\Psi(s_1)\Psi(s_2)\Psi(s_3)\Psi(s_4)) \\
 &\supseteq f_T(\Psi(s_2)) \cap f_T(\Psi(s_4)) \\
 &= (\Psi^{-1}(f_T))(s_2) \cap (\Psi^{-1}(f_T))(s_4)
 \end{aligned}$$

Thus, $\Psi^{-1}(f_T)$ is an S-int L-Q| ideal of S .

Theorem 3.77. Let f_S and f_T be SSs over U , and Ψ be a semigroup isomorphism from S to T . If f_T is an S-int Q| ideal of T , then $\Psi^{-1}(f_T)$ is an S-int Q| ideal of S .

Proposition 3.78. Let $f_S \in S_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $f_S = \tilde{S} \circ f_S \circ \tilde{S} \circ f_S$ for every S-int L-Q| ideal of S .

Proof: First assume that (1) holds. Let S be a regular semigroup, f_S be an S-int L-Q| ideal of S and $x \in S$. Then, $\tilde{S} \circ f_S \circ \tilde{S} \circ f_S \subseteq f_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\begin{aligned}
 (\tilde{S} \circ f_S \circ \tilde{S} \circ f_S)(x) &= \bigcup_{x=ab} \{(\tilde{S} \circ f_S \circ \tilde{S})(a) \cap f_S(b)\} \\
 &\supseteq (\tilde{S} \circ f_S \circ \tilde{S})(xy) \cap f_S(x) \\
 &= \left[\bigcup_{xy=ab} \{(\tilde{S} \circ f_S)(a) \cap \tilde{S}(b)\} \right] \\
 &\cap_{f_S(x)} \\
 &\supseteq \{(\tilde{S} \circ f_S)(x) \cap \tilde{S}(yxy)\} \cap_{f_S(x)} \\
 &= \left[\bigcup_{x=pq} \tilde{S}(p) \cap_{f_S(q)} \right] \cap U \cap_{f_S(x)}
 \end{aligned}$$

$$\begin{aligned}
 &\supseteq \{\tilde{S}(xy) \cap_{f_S(x)}\} \cap_{f_S(x)} \\
 &= U \cap_{f_S(x)} \cap_{f_S(x)} \\
 &= f_S(x)
 \end{aligned}$$

Therefore, $f_S \subseteq \tilde{S} \circ f_S \circ \tilde{S} \circ f_S$ implying that $f_S = \tilde{S} \circ f_S \circ \tilde{S} \circ f_S$.

Conversely, let $f_S = \tilde{S} \circ f_S \circ \tilde{S} \circ f_S$, where f_S is an S-int L-Q| ideal of S . In order to show that S is regular, we need to show that $A = SASA$ for every L-Q| ideal A of S . It is obvious that $SASA \subseteq A$. Thus, it is enough to show that $A \subseteq SASA$. Let $a \in A$, and A be any L-Q| ideal of S . Then, by Theorem 2.17, S_A is an S-int L-Q| ideal of S . By assumption and by Theorem 2.12, $S_A(a) = (\tilde{S} \circ S_A \circ \tilde{S} \circ S_A)(a) = (S_S \circ S_A \circ S_S \circ S_A)(a) = S_{SASA}(a) = U$ implying that $a \in SASA$. Hence, $A = SASA$, so S is a regular semigroup.

Proposition 3.79. Let $f_S \in S_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $f_S = f_S \circ \tilde{S} \circ f_S \circ \tilde{S}$ for every S-int R-Q| ideal of S .

Proof: First assume that (1) holds. Let S be a regular semigroup, f_S be an S-int R-Q| ideal of S and $x \in S$. Then, $f_S \circ \tilde{S} \circ f_S \circ \tilde{S} \subseteq f_S$ and there exists an element $y \in S$ such that $x = xyx$. Thus,

$$\begin{aligned}
 (f_S \circ \tilde{S} \circ f_S \circ \tilde{S})(x) &= \bigcup_{x=ab} \{f_S(a) \cap (\tilde{S} \circ f_S \circ \tilde{S})(b)\} \\
 &\supseteq f_S(x) \cap (\tilde{S} \circ f_S \circ \tilde{S})(yx) \\
 &= f_S(x) \\
 &\cap \left[\bigcup_{yx=ab} \{ \tilde{S}(a) \cap (f_S \circ \tilde{S})(b) \} \right] \\
 &\supseteq f_S(x) \cap \tilde{S}(yxy) \cap (f_S \circ \tilde{S})(x) \\
 &= f_S(x) \cap U \cap \left[\bigcup_{x=pq} f_S(p) \cap \tilde{S}(q) \right] \\
 &\supseteq f_S(x) \cap \{f_S(x) \cap \tilde{S}(yx)\} \\
 &= f_S(x) \cap_{f_S(x)} \cap U \\
 &= f_S(x)
 \end{aligned}$$

Therefore, $f_S \subseteq f_S \circ \tilde{S} \circ f_S \circ \tilde{S}$ implying that $f_S = f_S \circ \tilde{S} \circ f_S \circ \tilde{S}$.

Conversely, let $f_S = f_S \circ \tilde{S} \circ f_S \circ \tilde{S}$, where f_S is an S-int R-Q| ideal of S . In order to show that S is regular, we need to show that $A = ASAS$ for every R-Q| ideal A of S . It is obvious that $ASAS \subseteq A$. Thus, it is enough to show that $A \subseteq ASAS$. Let $a \in A$, and A be any R-Q| ideal of S . Then, by Theorem 2.17, S_A is an S-int R-Q| ideal of S .

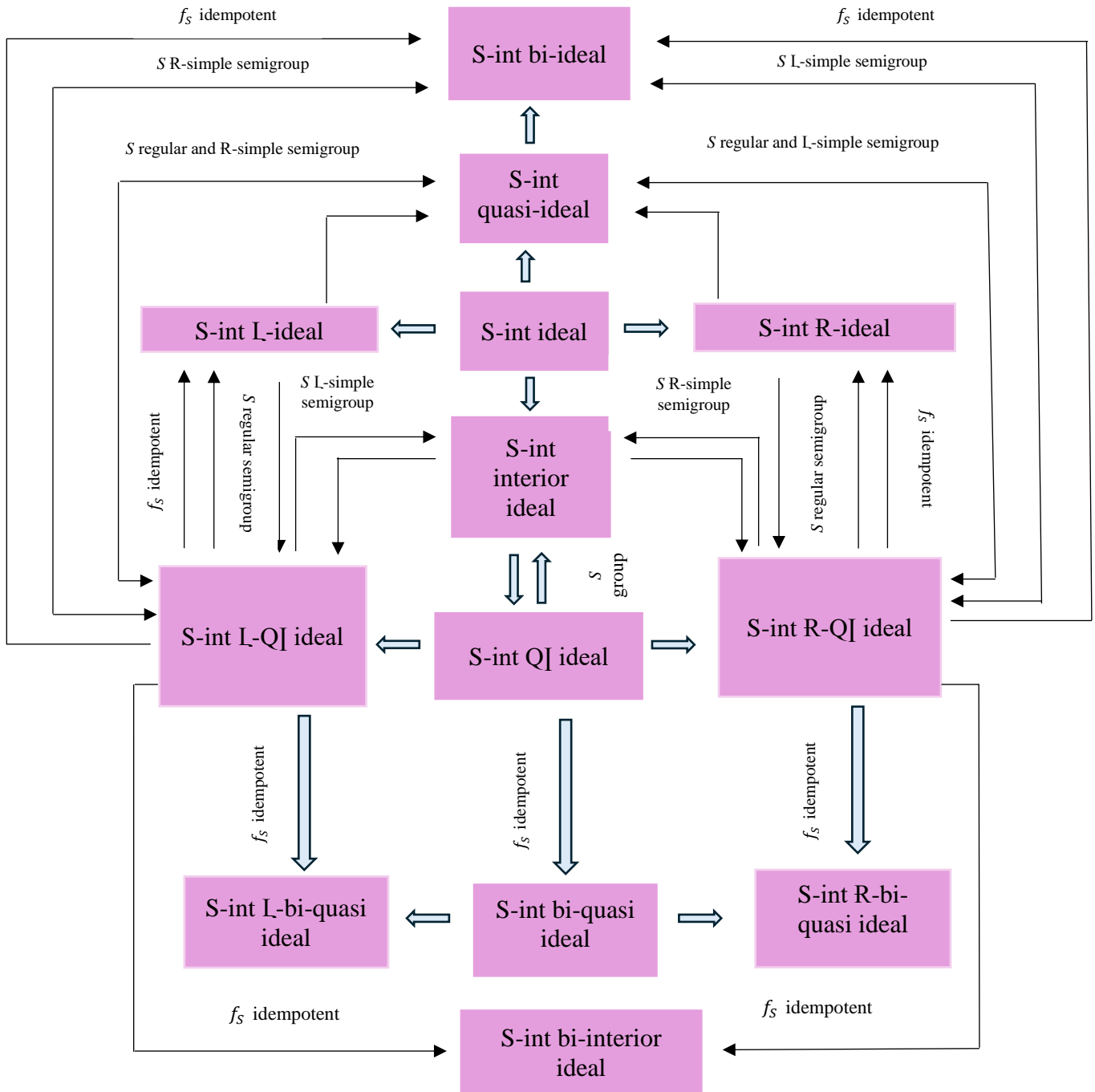
By assumption and by Theorem 2.12, $S_A(a) = (S_A \circ \tilde{S} \circ S_A \circ \tilde{S})(a) = (S_A \circ S_S \circ S_A \circ S_S)(a) = S_{ASAS}(a) = U$ implying that $a \in ASAS$. Hence, $A = ASAS$, so S is a regular semigroup.

Theorem 3.80. Let $f_S \in S_S(U)$. Then, the following conditions are equivalent:

1. S is regular.
2. $f_S = \tilde{S} \circ_{f_S} \tilde{S} \circ_{f_S} f_S = f_S \circ \tilde{S} \circ_{f_S} \tilde{S}$ for every S-int QI ideal of S .

4. CONCLUSION

As a generalization of quasi ideals and interior ideals of semigroups, Rao [9] introduced the concept of QI



The relation between several S-int ideals and their generalized ideals is depicted in the following figure, where $A \rightarrow B$ denotes that $A \subseteq B$ but B may not always be A .

ideals and investigated their properties in semigroups. In this study, we introduced "S-int Q| ideals of semigroups" and applied this concept to both SS theory and semigroup theory. We derived the relationships between S-int Q| ideals and various types of S-int ideals in a semigroup. We demonstrated that an S-int ideal and an S-int interior ideal are both S-int Q| ideals. Additionally, the S-int bi-ideal of a group is an S-int Q| ideal, the S-int quasi-ideal of a regular group is an S-int Q| ideal, the idempotent S-int Q| ideal is an S-int bi-quasi-ideal, and an S-int bi-interior ideal. However, the converse statements are not true, as shown by counterexamples. For the converse statements, we demonstrate that the semigroup should be either a group or regular, and the S-int Q| ideal should be idempotent. Using our key theorem, we showed that if a subsemigroup of a semigroup is a Q| ideal, then its $\square CF$ is an S-int Q| ideal, and vice versa. We demonstrated how this concept is connected to the existing algebraic structures in classical semigroup theory. In doing so, we established a clear relationship between semigroup theory and SS theory. Additionally, we provided conceptual characterizations and an analysis of the novel idea in the context of soft image, soft inverse image, and SS operations, supporting our claims with specific and insightful examples. Furthermore, the connection between a regular semigroup and the structure of S-int Q| ideals was established and demonstrated. In future studies, different S-int ideals of semigroups can be examined, and additionally, the S-int Q| ideals introduced in this work can be characterized with intra-regular, zero semigroups, and semisimple semigroups.

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