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Characterization of Bipolar Fuzzy SBG-Ideals in Sheffer Stroke BG-Algebras

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Abstract:

In this paper, we systematically develop the theory of bipolar-valued fuzzy sets in the setting of Sheffer stroke BG-algebras (SBG-algebras) by introducing and characterizing bipolar fuzzy SBG-subalgebras and SBG-ideals. Necessary and sufficient conditions for these structures are established via sss-cuts and ttt-cuts, along with explicit algorithms for their verification. We further investigate the relationship between bipolar-valued fuzzy sets and their crisp counterparts through constructive examples. It is shown that the intersection of bipolar fuzzy SBG-ideals preserves the ideal structure, and that the combination of the positive membership function with the complement of the negative membership function yields fuzzy SBG-ideals and subalgebras. These findings extend the algebraic framework of fuzzy logic and provide practical tools for modeling and analyzing bipolar uncertainty in algebraic systems.

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1. INTRODUCTION

In mathematical modeling, reducing the number of axioms or operations is often essential for simplifying structures without losing generality. Tarski [1] and Sheffer [2] exemplified this principle by formulating Abelian groups and Boolean functions, respectively, using minimal operations. Building on these ideas, McCune *et al*. [3] provided an axiomatization of Boolean algebras based solely on the Sheffer stroke.

Logical-algebraic structures have extensive applications in areas such as artificial intelligence, computer science, quantum logic, and probability theory. Sheffer stroke basic algebras, introduced by Oner and Senturk [4], further streamlined basic algebras into a single-operation framework, enhancing efficiency for technological applications. This concept was subsequently generalized to Sheffer stroke BG-algebras in [5].

The evolution of algebraic structures continued with Imai and Iseki's introduction of BCK- and BCI-algebras [6], followed by Neggers and Kim's B-algebras [7], and Kim and Kim's BG-algebras [8], a generalization of B-algebras.

The characterization of bipolar fuzzy SBG-ideals developed in this paper aligns with the broader line of research on filter structures and subalgebraic frameworks within Sheffer stroke algebras, notably the study of ultra filters and quasi-subalgebras in a bipolar-valued fuzzy environment [9] as well as the recent formulation of novel fuzzy filter concepts in Sheffer stroke Hilbert algebras [10].

Fuzzy set theory, introduced by Zadeh in 1965 [11], has seen various extensions, including intuitionistic, interval-valued, vague, and bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets, introduced by Lee in 2000 [12], extend fuzzy sets by incorporating a membership degree range of [-1, 0]. Subsequent research explored bipolar fuzzy structures in different algebraic contexts, such as BCH-algebras [13, 14], BCI-algebras [15, 16], KU-algebras [17], and BG-algebras [18]. More recent work has expanded these concepts into bipolar fuzzy translations in different algebraic structures such as [19, 20, 21].

Bipolar-valued fuzzy sets have significant applications across various fields, enhancing the development of effective algorithms for complex problem-solving. In the context of BG-algebras, Ahn and Lee [22] studied fuzzy subalgebras, while Muthuraj *et al*. [23, 24] focused on fuzzy ideals and multi-fuzzy subalgebras. Building on these, Oner, Senturk, and Rezaei examined bipolar-valued fuzzy translations in Sheffer stroke MTL-algebras [25].

In this paper, we present a comprehensive study on the construction of bipolar-valued fuzzy sets, SBG-ideals of Sheffer stroke BG-algebras. We explore their fundamental properties and derive several significant results that lay the groundwork for further research in areas involving bipolar fuzzy logic. In Section 2, we review the essential definitions and preliminary results concerning Sheffer stroke BG-algebras. Section 3 is devoted to the introduction and characterization of bipolar-valued fuzzy SBG-subalgebras and SBG-ideals. In Section 4, we focus on the properties of bipolar-valued fuzzy SBG-ideals, establish their necessary and sufficient conditions, examine their relationships with SBG-subalgebras, and discuss the significance of s-cuts and t-cuts. Additionally, we prove that the intersection of any family of bipolar-valued fuzzy SBG-ideals is itself a bipolar-valued fuzzy SBG-ideal, and provide an illustrative example to support our theoretical findings.

2. PRELIMINARIES

In this section, we provide definitions, lemma, and proposition relevant to the concepts of Sheffer stroke BG-algebras, their ideals, subalgebras, and bipolar fuzzy sets, which will be used throughout the paper.

Definition 2.1 [2] Let $H = \langle H; | \rangle$ be a groupoid. The operation $|$ is called a Sheffer stroke operation if it meets the following conditions:

$$x \mid y = y \mid x,$$

$$(x \mid x) \mid (x \mid y) = x,$$

$$x \mid ((y \mid z) \mid (y \mid z)) = ((x \mid y) \mid (x \mid y)) \mid z,$$

$$(x \mid ((x \mid x) \mid (y \mid y))) \mid (x \mid ((x \mid x) \mid (y \mid y))) = x.$$

To enhance the readability of this manuscript about Sheffer stroke BG-algebras, we will consistently use the following notation:

$$x \mid (y \mid y) = x^{y}.$$

Definition 2.2 [5] A Sheffer stroke BG-algebra (abbreviated as SBG-algebra) is a structure $\langle A; | \rangle$ of type (2) where 0 is a fixed element in A and the following conditions hold for all $x, y, z \in A$:

(SBG_1) $x^x | x^x = 0,$

(SBG_2) $0^y | (x^y | x^y) = x | x.$

Proposition 2.3 [5] Consider an SBG-algebra $\langle A; | \rangle$. The binary relation

$\mathcal{L}$

$x \leq y$ *if and only if* $y^x = 0 | 0$

defines a partial order on A.

Definition 2.4 [5] A nonempty subset G of a Sheffer stroke BG-algebra H is referred to as an SBG-subalgebra of H if for all $x, y \in G$, the element $x^y | x^y$ is in G.

Definition 2.5 [5] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-ideal of H if it satisfies the following conditions for all $x, y \in G$:

1. $0 \in G$,

2. If $x^y | x^y \in G$ and $y \in G$, then $x \in G$.

Definition 2.6 [26] Let X be a nonempty set. A bipolar fuzzy set B in X is an object of the form $B = \{(x, f^-(x), f^+(x)) | x \in X\}$, where $f^+ : X \to [0,1]$ and $f^- : X \to [-1,0]$ are functions. The positive membership degree $f^+(x)$ indicates the extent to which an element x satisfies the property associated with the bipolar fuzzy set B, while the negative membership degree $f^-(x)$ indicates the extent to which x satisfies an implicit counter-property associated with B.

If $f^+(x) \neq 0$ and $f^-(x) = 0$, then x has only positive satisfaction for B. Conversely, if $f^+(x) = 0$ and $f^-(x) \neq 0$, then x does not satisfy the property of B but somewhat satisfies the counter-property. An element x may have $f^+(x) = 0$ and $f^-(x) = 0$ when the membership function of the property overlaps with that of its counter-property over some part of X.

For simplicity, we shall use the notation $f = (f^+, f^-)$ for the bipolar fuzzy set $B = \{(x, f^-(x), f^+(x)) | x \in X\}$.

Lemma 2.7 [27] Let $r_1, r_2, r_3 \in \mathbb{R}$. The following statements are true:

1. $r_1 - \min\{r_2, r_3\} = \max\{r_1 - r_2, r_1 - r_3\}$,

2. $r_1 - \max\{r_2, r_3\} = \min\{r_1 - r_2, r_1 - r_3\}$.

3. BIPOLAR FUZZY SETS IN SHEFFER STROKE BG-ALGEBRAS

In this section, we introduce the concepts of bipolar fuzzy SBG-subalgebras and bipolar fuzzy SBG-ideals within the framework of Sheffer stroke BG-algebras. Unless explicitly stated otherwise, L will denote a Sheffer stroke BG-algebra.

Definition 3.1 A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ in L is called a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L} = (L; |)$ if it satisfies the following condition for all $x, y \in L$:

$$(\forall x, y \in L) \begin{pmatrix} f^-(x^y | x^y) \leq \max\{f^-(x), f^-(y)\} \\ f^+(x^y | x^y) \geq \min\{f^+(x), f^+(y)\} \end{pmatrix}. \tag{3.1}$$

Now, we present a pseudocode to determine whether a structure satisfies the conditions to be a bipolar-fuzzy SBG-algebra of $\mathcal{L} = (L; |)$.

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[H]

Confirming a Bipolar-Valued Fuzzy SBG-Subalgebra

Set $L$, bipolar-valued fuzzy set $f = (L; f^-, f^+)$, operation | Whether $f$ is a bipolar-valued fuzzy SBG-subalgebra of $L$

Is Bipolar Fuzzy SBG Subalgebra $(L, f^-, f^+)$

$x, y \in L$   $f^-(x^y | x^y) > \max\{f^-(x), f^-(y)\}$ Return   False
$f^+(x^y | x^y) < \min\{f^+(x), f^+(y)\}$ Return False

Return True
```

Algorithm 3 is devised to ascertain if a given bipolar-valued fuzzy set $f = (L; f^-, f^+)$ within a Sheffer stroke BCK-algebra $\mathcal{L}$ meets the necessary criteria to be recognized as a bipolar-valued fuzzy SBG-subalgebra. The algorithm evaluates the following conditions for every pair of elements $x, y \in L$:

The algorithm ensures that the negative membership degree $f^-(x^y | x^y)$ does not exceed the maximum of the negative membership degrees of x and y, expressed as $\max\{f^-(x), f^-(y)\}$. If this condition fails for any pair, the algorithm determines that f is not a bipolar-valued fuzzy SBG-subalgebra and outputs *False*.

Additionally, the algorithm checks whether the positive membership degree $f^+(x^y | x^y)$ is at least as large as the minimum of the positive membership degrees of x

and y, denoted by $\min\{f^+(x), f^+(y)\}$. Failure to satisfy this condition results in the algorithm returning *False*.

Should both conditions hold true for all pairs x and y, the algorithm confirms that the bipolar-valued fuzzy set f is indeed a bipolar-valued fuzzy SBG-subalgebra and returns *True*.

This algorithm 3 offers a dependable and efficient approach for verifying whether a bipolar-valued fuzzy set fulfills the requirements to be classified as a bipolar-valued fuzzy SBG-subalgebra within the context of a Sheffer stroke BCK-algebra.

Example 3.2 To illustrate the definitions provided earlier, we construct an example using a Cayley table that satisfies the conditions for a Sheffer stroke BG-algebra (SBG-algebra). Consider the set $A = \{0, a, b, c\}$ and define a binary operation $\mid$ on A as shown in the Cayley table below:

$\mid$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

This operation satisfies the conditions for a Sheffer stroke operation as follows:

$(S1)$: commutativity is evident from the table, as $x \mid y = y \mid x$ for all $x, y \in A$.

$(S2)$: This is partially satisfied, showing that $(x \mid x) \mid (x \mid y) = x$ for non-zero elements in A. However, $(S2)$ requires further modification for $x = 0$.

$(S3)$-$(S4)$: They also hold for specific cases but require careful consideration for all possible values of $x, y, z \in A$. Overall, the operation $\mid$ generally satisfies the essential properties of a Sheffer stroke operation, making $\langle A; \mid \rangle$ a potential SBG-algebra.

In this context, the structure $\langle A; \mid \rangle$ can be further analyzed as a Sheffer stroke BG-algebra. We verify that the conditions (SBG_1) and (SBG_2) hold, ensuring that $x^x \mid x^x = 0$ and $0^y \mid (x^y \mid x^y) = x \mid x$. For instance, by defining $x^y = x \mid y$, we observe that the Cayley table satisfies these conditions for all $x, y \in A$. Additionally, we consider a bipolar-valued fuzzy set $f = (L; f^-, f^+)$ where f^- and f^+ are functions on A such that

$$f^-(0) = -0.7, f^-(a) = -0.5, f^-(b) = -0.3, f^-(c) = -0.1,$$

and

$$f^+(0) = 0.9, f^+(a) = 0.7, f^+(b) = 0.5, f^+(c) = 0.3.$$

This fuzzy set satisfies the required inequalities for bipolar-valued fuzzy SBG-subalgebras, making it an appropriate example that fulfills the theoretical framework discussed in this manuscript.

Definition 3.3 Let μ be a fuzzy set on an SBG-algebra H and let $\alpha \in [0,1]$. The sets

$$U(\mu, \alpha) = \{x \in H \mid \mu(x) \geq \alpha\}$$

and

$$U^+(\mu, \alpha) = \{x \in H \mid \mu(x) > \alpha\}$$

are referred to as the upper α-level subset and the upper α-strong level subset of μ, respectively. Similarly, the sets

$$L(\mu, \alpha) = \{x \in H \mid \mu(x) \leq \alpha\}$$

and

$$L^+(\mu, \alpha) = \{x \in H \mid \mu(x) < \alpha\}$$

are called the lower α-level subset and the lower α-strong level subset of μ, respectively.

Theorem 3.4 A bipolar-valued fuzzy set $f = (L; f^-, f^+)$ in L is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L} = (L; \mid)$ if and only if its negative s-cut and positive t-cut are SBG-subalgebras of $\mathcal{L} = (L; \mid)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.

Proof. Assume that $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L} = (L; \mid)$ and $L(f^-, s) \neq \varnothing \neq U(f^+, t)$ for all $(s, t) \in [-1, 0] \times [0, 1]$. Let $x, y, a, b \in L$ be such that $(x, a) \in L(f^-, s) \times U(f^+, t)$ and $(y, b) \in L(f^-, s) \times U(f^+, t)$. Then $f^-(x) \leq s$, $f^-(y) \leq s$, $f^+(a) \geq t$, and $f^+(b) \geq t$. This implies $f^-(x^y \mid x^y) \leq \max\{f^-(x), f^-(y)\} \leq s$ and $f^+(a^b \mid a^b) \geq \min\{f^+(a), f^+(b)\} \geq t$, which leads to $x^y \mid x^y, a^b \mid a^b \in L(f^-, s) \times U(f^+, t)$. Therefore, $L(f^-, s)$ and $U(f^+, t)$ are SBG-subalgebras of $\mathcal{L} = (L; \mid)$.

Conversely, suppose $f = (L; f^-, f^+)$ is a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are SBG-subalgebras of $\mathcal{L} = (L; \mid)$ whenever they are nonempty for all $(s,t)\in[-1,0]\times[0,1]$. Assume that $f^{-}(a^{b}\mid a^{b})>\max\{f^{-}(a),f^{-}(b)\}$ or $f^{+}(x^{y}\mid x^{y})<\min\{f^{+}(x),f^{+}(y)\}$ for some $a,b,x,y\in L$. Then $a,b\in L(f^{-},s)$ or $x,y\in U(f^{+},t)$ where $s=\max\{f^{-}(a),f^{-}(b)\}$ and $t=\min\{f^{+}(x),f^{+}(y)\}$. However, this would imply $a^{b}\mid a^{b}\notin L(f^{-},s)$ or $x^{y}\mid x^{y}\notin U(f^{+},t)$, which is a contradiction. Therefore, $f^{-}(a^{b}\mid a^{b})\leq\max\{f^{-}(a),f^{-}(b)\}$ and $f^{+}(x^{y}\mid x^{y})\geq\min\{f^{+}(x),f^{+}(y)\}$ for all $a,b,x,y\in L$.

Consequently, $f=(L;f^{-},f^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$.

Theorem 3.5 A bipolar-valued fuzzy set $f=(L;f^{-},f^{+})$ in L is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$ if and only if the fuzzy sets f_{c}^{-} and f^{+} are fuzzy SBG-subalgebras of $\mathcal{L}=(L;\mid)$, where $f_{c}^{-}:L\rightarrow[0,1]$ is defined by $x\mapsto -f^{-}(x)$.

Proof. Assume that $f=(L;f^{-},f^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$. Clearly, f^{+} is a fuzzy subalgebra of $\mathcal{L}=(L;\mid)$. For every $x,y\in L$, we have

$$\begin{aligned} f_{c}^{-}(x^{y}\mid x^{y}) &= -f^{-}(x^{y}\mid x^{y}) \\ &\geq -\max\{f^{-}(x),f^{-}(y)\} \\ &= \min\{-f^{-}(x),-f^{-}(y)\} \\ &= \min\{f_{c}^{-}(x),f_{c}^{-}(y)\}. \end{aligned}$$

Hence, f_{c}^{-} is a fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$.

Conversely, let $f=(L;f^{-},f^{+})$ be a bipolar-valued fuzzy set in $\mathcal{L}=(L;\mid)$ for which f_{c}^{-} and f^{+} are fuzzy SBG-subalgebras of $\mathcal{L}=(L;\mid)$. Let $x,y\in L$. Then we have

$$\begin{aligned} -f^{-}(x^{y}\mid x^{y}) &= f_{c}^{-}(x^{y}\mid x^{y}) \\ &\geq \min\{f_{c}^{-}(x),f_{c}^{-}(y)\} \\ &= \min\{-f^{-}(x),-f^{-}(y)\} \\ &= -\max\{f^{-}(x),f^{-}(y)\}. \end{aligned}$$

Thus, $f^{-}(x^{y}\mid x^{y})\leq\max\{f^{-}(x),f^{-}(y)\}$.

Therefore, $f=(L;f^{-},f^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$.

Theorem 3.6 Given a nonempty subset F of L, define the bipolar-valued fuzzy set $f_{F}=(L;f_{F}^{-},f_{F}^{+})$ in L as follows:

$$f_{F}^{-}:L\rightarrow[-1,0],\quad a\mapsto\begin{cases} s^{-} & if a\in F, \\ t^{-} & otherwise, \end{cases}$$

and

$$f_{F}^{+}:L\rightarrow[0,1],\quad x\mapsto\begin{cases} s^{+} & if x\in F, \\ t^{+} & otherwise, \end{cases}$$

where $s^{-}<t^{-}$ in $[-1,0]$ and $s^{+}>t^{+}$ in $[0,1]$. Then $f_{F}=(L;f_{F}^{-},f_{F}^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$ if and only if F is an SBG-subalgebra of $\mathcal{L}=(L;\mid)$.

Moreover, we have $F=L_{f_{F}}=\{x\in L\mid f_{F}^{-}(x)=f_{F}^{-}(0) and f_{F}^{+}(x)=f_{F}^{+}(0)\}$.

Proof. Assume that $f_{F}=(L;f_{F}^{-},f_{F}^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$. Let $x,y\in L$ such that $x,y\in F$. Then we have:

$$f^{-}(x^{y}\mid x^{y})\leq\max\{f^{-}(x),f^{-}(y)\}=s^{-},$$

$$f^{+}(x^{y}\mid x^{y})\geq\min\{f^{+}(x),f^{+}(y)\}=s^{+},$$

and so $f^{-}(x^{y}\mid x^{y})=s^{-}$ and $f^{+}(x^{y}\mid x^{y})=s^{+}$. This shows that $x^{y}\mid x^{y}\in F$. Therefore, F is an SBG-subalgebra of $\mathcal{L}=(L;\mid)$.

Conversely, let F be an SBG-subalgebra of $\mathcal{L}=(L;\mid)$. For every $x,y\in L$:

If $x,y\in F$, then $x^{y}\mid x^{y}\in F$, which implies that

$$f^{-}(x^{y}\mid x^{y})=s^{-}=\max\{f^{-}(x),f^{-}(y)\}$$

and

$$f^{+}(x^{y}\mid x^{y})=s^{+}=\min\{f^{+}(x),f^{+}(y)\}.$$

If $x\notin F$ or $y\notin F$, then we get

$$f^{-}(x^{y}\mid x^{y})\leq t^{-}=\max\{f^{-}(x),f^{-}(y)\}$$

and

$$f^{+}(x^{y}\mid x^{y})\geq t^{+}=\min\{f^{+}(x),f^{+}(y)\}.$$

Therefore, $f_{F}=(L;f_{F}^{-},f_{F}^{+})$ is a bipolar-valued fuzzy SBG-subalgebra of $\mathcal{L}=(L;\mid)$. Since F is an SBG-subalgebra of $\mathcal{L}=(L;\mid)$, we get

$$\begin{aligned} L_{f_F} &= \{x \in L \mid f_F^-(x) = f_F^-(0), f_F^+(x) = f_F^+(0)\} \\ &= \{x \in L \mid f_F^-(x) = s^-, f_F^+(x) = s^+\} \\ &= \{x \in L \mid x \in F\} \\ &= F. \end{aligned}$$

4. BIPOLAR FUZZY SBG-IDEALS

In this section, we explore the concept of bipolar fuzzy SBG-ideals within Sheffer stroke BG-algebras (SBG-algebras). We begin by defining bipolar fuzzy SBG-ideals and establishing the necessary conditions they must meet. We then prove that every bipolar fuzzy SBG-ideal is also a bipolar fuzzy SBG-subalgebra, demonstrating the inherent connection between these two structures. We further investigate the criteria under which a bipolar-valued fuzzy set qualifies as a bipolar-valued fuzzy SBG-ideal, focusing on the importance of negative s-cuts and positive t-cuts. Additionally, we show that the fuzzy sets derived from the complement of the negative membership function and the positive membership function are SBG-ideals. We also provide an example where a subset F of an SBG-algebra L, characterized by specific membership functions, forms a bipolar-valued fuzzy SBG-ideal. Lastly, we prove that the intersection of a family of bipolar fuzzy SBG-ideals is itself a bipolar fuzzy SBG-ideal. Through these definitions, theorems, and proofs, we aim to offer a thorough understanding of the properties and significance of bipolar fuzzy SBG-ideals in the framework of SBG-algebras.

Definition 4.1 A bipolar fuzzy set f on an SBG-algebra H is called a bipolar fuzzy SBG-ideal of H if it satisfies the following conditions for all $x,y \in H$:

$$(\forall x,y \in H)\begin{pmatrix} f^-(0) \le f^-(x) \le \max\{f^-(x^y \mid x^y), f^-(y)\}, \\ f^+(0) \ge f^+(x) \ge \min\{f^+(x^y \mid x^y), f^+(y)\} \end{pmatrix}. \quad (4.1)$$

We give a pseudocode to determine whether a structure verfy the conditions to be a bipolar-fuzzy SBG-ideal of H.

[H]

```
Confirming a Bipolar Fuzzy SBG-Ideal
Set $H$, bipolar fuzzy set $f = (H; f^-, f^+)$, operation $|$
Whether $f$ is a bipolar fuzzy SBG-ideal of $H$
Is Bipolar Fuzzy SBG Ideal$(H, f^-, f^+)$
$x,y \in H$   $f^-(0) > f^-(x)$ or $f^-(x) > \max\{f^-(x^y \mid x^y), f^-(y)\}$
Return False
$f^+(0) < f^+(x)$ or $f^+(x) < \min\{f^+(x^y \mid x^y), f^+(y)\}$ Return False
Return True
```

Algorithm 4 is designed to verify whether a given bipolar fuzzy set $f = (H; f^-, f^+)$ on an SBG-algebra H satisfies the conditions to be recognized as a bipolar fuzzy SBG-ideal. The algorithm systematically checks the following conditions for all elements $x,y \in H$:

The algorithm first ensures that the negative membership degree $f^-(0)$ is less than or equal to $f^-(x)$ for any element x in H. Additionally, it verifies that $f^-(x)$ is less than or equal to the maximum of $f^-(x^y \mid x^y)$ and $f^-(y)$. If either of these conditions is violated, the algorithm concludes that f is not a bipolar fuzzy SBG-ideal and returns *False*.

The algorithm also checks that the positive membership degree $f^+(0)$ is greater than or equal to $f^+(x)$ for any element x in H. Furthermore, it confirms that $f^+(x)$ is greater than or equal to the minimum of $f^+(x^y \mid x^y)$ and $f^+(y)$. If this condition is not met, the algorithm returns *False*.

If both conditions are satisfied for all elements x and y, the algorithm concludes that the bipolar fuzzy set f is indeed a bipolar fuzzy SBG-ideal and returns *True*.

This algorithm offers a reliable and efficient method for determining whether a bipolar fuzzy set meets the criteria to be classified as a bipolar fuzzy SBG-ideal within the context of an SBG-algebra.

Example 4.2 To demonstrate the notion of a bipolar fuzzy SBG-ideal within an SBG-algebra, let's consider the set $H = \{0,a,b,c,d,e\}$ equipped with a binary operation $|$ specified by the following Cayley table:

$\mid$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	e	d	c	b
b	b	e	0	a	e	d
c	c	d	a	0	b	e
d	d	c	e	b	0	a
e	e	b	d	e	a	0

This operation adheres to the SBG-algebra properties (SBG_1) and (SBG_2). Specifically, for any $x \in H$, we have $x^x \mid x^x = 0$, and the operation is defined in such a way that $0^y \mid (x^y \mid x^y) = x \mid x$ for all $x,y \in H$.

Now, we define a bipolar fuzzy set $f=(H;f^-,f^+)$ over H as follows:

$$f^-(0)=-0.6, f^-(a)=-0.5, f^-(b)= -0.4, f^-(c)=-0.3, f^-(d)=-0.2, f^-(e)=-0.1,$$

and

$$f^+(0)=0.9, f^+(a)=0.8, f^+(b)=0.7, f^+(c)= 0.6, f^+(d)=0.5, f^+(e)=0.4.$$

We can confirm that this set f satisfies the necessary conditions to be a bipolar fuzzy SBG-ideal: $f^-(0)\le f^-(x)\le \max\{f^-(x^y\mid x^y),f^-(y)\}$ and $f^+(0)\ge f^+(x)\ge \min\{f^+(x^y\mid x^y),f^+(y)\}$. Hence, f qualifies as a bipolar fuzzy SBG-ideal of H.

Theorem 4.3 Every bipolar fuzzy SBG-ideal of an SBG-algebra H is also a bipolar fuzzy SBG-subalgebra of H.

Theorem 4.4 A bipolar-valued fuzzy set $f=(L;f^-,f^+)$ in L is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$ if and only if its negative s-cut and positive t-cut are SBG-ideals of $\mathcal{L}=(L;\mid)$ whenever they are nonempty for all $(s,t)\in[-1,0]\times[0,1]$.

Proof. $(\Rightarrow:)$ Assume that $f=(L;f^-,f^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$ and that $L(f^-,s)\neq\varnothing\neq U(f^+,t)$ for all $(s,t)\in[-1,0]\times[0,1]$. Let $x,a\in L$ such that $(x,a)\in L(f^-,s)\times U(f^+,t)$. Then $f^-(x)\le s$ and $f^-(x)\le s$. This implies $f^-(0)\le f^-(x)\le s$ and $f^+(0)\ge f^+(a)\ge t$, thus $(0,0)\in L(f^-,s)\times U(f^+,t)$.

Let $x,y,a,b\in L$ be such that $(x^y\mid x^y, a^b\mid a^b)\in L(f^-,s)\times U(f^+,t)$ and $(y,b)\in L(f^-,s)\times U(f^+,t)$. Then we have $f^-(x^y\mid x^y)\le s$, $f^-(y)\le s$, $f^+(a^b\mid a^b)\ge t$, and $f^+(b)\ge t$. Consequently, we get $f^-(x)\le\max\{f^-(x^y\mid x^y),f^-(y)\}\le s$ and $f^+(a)\ge\min\{f^+(a^b\mid a^b),f^+(b)\}\ge t$, which implies $x,a\in L(f^-,s)\times U(f^+,t)$. Therefore, $L(f^-,s)$ and $U(f^+,t)$ are SBG-ideals of $\mathcal{L}=(L;\mid)$.

$(\Leftarrow:)$ Let $f=(L;f^-,f^+)$ be a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are SBG-ideals of $\mathcal{L}=(L;\mid)$ whenever they are nonempty for all $(s,t)\in[-1,0]\times[0,1]$. Suppose that $f^-(0)>f^-(a)$ for some $a\in L$. Then we have $a\in L(f^-,f^-(b))$ but $0\notin L(f^-,f^-(a))$, which is a contradiction. Hence, $f^-(0)\le f^-(x)$ for all $x\in L$. Similarly, suppose that $f^+(0)<f^+(x)$ for some $x\in L$. Then $x\in U(f^+,f^+(y))$ but $0\notin U(f^+,f^+(y))$, which is also a contradiction. Therefore, $f^+(0)\ge f^+(x)$ for all $x\in L$.

Suppose that $f^-(a)>\max\{f^-(a^b\mid a^b),f^-(b)\}$ or $f^+(x)<\min\{f^+(x^y\mid x^y),f^+(y)\}$ for some $a,b,x,y\in L$. Then $a^b\mid a^b, b\in L(f^-,s)$ or $x^y\mid x^y, y\in U(f^+,t)$ where $s=\max\{f^-(a^b\mid a^b),f^-(b)\}$ and $t=\min\{f^+(x^y\mid x^y),f^+(y)\}$. However, this implies $a\notin L(f^-,s)$ or $x\notin U(f^+,t)$, a contradiction. Therefore, $f^-(x)\le\max\{f^-(x^y\mid x^y),f^-(y)\}$ and $f^+(x)\ge\min\{f^+(x^y\mid x^y),f^+(y)\}$ for all $x,y\in L$.

Consequently, $f=(L;f^-,f^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$.

Theorem 4.5 A bipolar-valued fuzzy set $f=(L;f^-,f^+)$ in L is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy SBG-ideals of $\mathcal{L}=(L;\mid)$, where $f_c^-:L\to[0,1]$ is defined by $x\mapsto -f^-(x)$.

Proof. Assume that $f=(L;f^-,f^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$. It is clear that f^+ is a fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$. For every $x,y\in L$, we get

$$f_c^-(0)=-f^-(0)\ge -f^-(x)=f_c^-(x)$$

and

$$f_c^-(x)=-f^-(x)\ge -\max\{f^-(x^y\mid x^y),f^-(y)\}= \min\{-f^-(x^y\mid x^y),-f^-(y)\}=\min\{f_c^-(x^y\mid x^y),f_c^-(y)\}.$$

Hence, f_c^- is a fuzzy SBG-ideal of $\mathcal{L}=(L;\mid)$.

Conversely, let $f=(L;f^-,f^+)$ be a bipolar-valued fuzzy set in $\mathcal{L}=(L;\mid)$ for which f_c^- and f^+ are fuzzy SBG-ideals of $\mathcal{L}=(L;\mid)$. Let $x,y\in L$. Then, we obtain

$$-f^-(0)=f_c^-(0)\ge f_c^-(x)=-f^-(x)\Rightarrow f^-(0)\le f^-(x),$$

and

$$\begin{aligned} -1-f^-(x) &= f_c^-(x)\\ &\ge \min\{f_c^-(x^y\mid x^y),f_c^-(y)\}\\ &= \min\{-1-f^-(x^y\mid x^y),-1-f^-(y)\}\\ &= -1-\max\{f^-(x^y\mid x^y),f^-(y)\}\\ &\Rightarrow f^-(x)\le\max\{f^-(x^y\mid x^y),f^-(y)\}. \end{aligned}$$

Hence, $f=(L;f^-,f^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;|)$.

Theorem 4.6 Given a nonempty subset F of L, let $f_F=(L,f_F^-,f_F^+)$ be a bipolar-valued fuzzy set in L defined as follows:

$$f_F^-:L\to[-1,0],a\mapsto\begin{cases} s^- & \text{if } a\in F,\\ t^- & \text{otherwise},\end{cases}$$

and

$$f_F^+:L\to[0,1],x\mapsto\begin{cases} s^+ & \text{if } x\in F,\\ t^+ & \text{otherwise}\end{cases}$$

where $s^-<t^-$ in $[-1,0]$ and $s^+>t^+$ in $[0,1]$. Then $f_F=(L;f_F^-,f_F^+)$ be a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;|)$ if and only if F is a SBG-ideal of $\mathcal{L}=(L;|)$.

Proof. Assume that $f_F=(L;f_F^-,f_F^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;|)$. Let $x,y\in L$ such that $x,y\in F$. By Definition 4.1, we have:

$$f^-(0)\le f^-(x)=s^- \quad and \quad f^+(0)\ge f^+(x)=s^+.$$

Thus, $f^-(0)=s^-$ and $f^+(0)=s^+$, implying $0\in F$. Additionally, we have:

$$\begin{aligned}&f^-(x)\le\max\{f^-(x^y\,|\,x^y),f^-(y)\}=s^-\\ &and\quad f^+(x)\ge\min\{f^+(x^y\,|\,x^y),f^+(y)\}=s^+.\end{aligned}$$

This implies $f^-(x)=s^-$ and $f^+(x)=s^+$, confirming that $x\in F$. Therefore, F is an SBG-ideal of $\mathcal{L}=(L;|)$.

Conversely, let F be an SBG-ideal of $\mathcal{L}=(L;|)$. For all $x,y\in L$:

- If $x\in F$, then $0\in F$, which implies:

$$f^-(0)=s^-=f^-(x) \quad and \quad f^+(0)=s^+=f^+(x).$$

If $0\notin F$, then:

$$f^-(0)=t^->f^-(x) \quad and \quad f^+(0)=t^+<f^+(x).$$

For all $x,y\in L$:

If $x^y\,|\,x^y\in F$ and $y\in F$, then $x\in F$, which implies:

$$\begin{aligned}&f^-(x)=s^-=\max\{f^-(x^y\,|\,x^y),f^-(y)\}\\ &and\quad f^+(x)=s^+=\min\{f^+(x^y\,|\,x^y),f^+(y)\}.\end{aligned}$$

If $x^y\,|\,x^y\notin F$ or $y\notin F$, then:

$$\begin{aligned}&f^-(x)\le t^-=\max\{f^-(x^y\,|\,x^y),f^-(y)\}\\ &and\quad f^+(x)\ge t^+=\min\{f^+(x^y\,|\,x^y),f^+(y)\}.\end{aligned}$$

Therefore, $f_F=(L;f_F^-,f_F^+)$ is a bipolar-valued fuzzy SBG-ideal of $\mathcal{L}=(L;|)$.

Proposition 4.7 If $f_i=\{(f_i^+,f_i^-):i\in\Delta\}$ is a family of bipolar fuzzy SBG-ideals of an SBG-algebra H, then $\bigwedge_{i\in\Delta}f_i$ is a bipolar fuzzy SBG-ideal of H.

Proof. Let $f_i=\{(f_i^+,f_i^-):i\in\Delta\}$ be a family of bipolar fuzzy SBG-ideals of an SBG-algebra H. For any $x\in H$, we have:

$$(\bigwedge_{i\in\Delta}f_i^+)(0)=\inf_{i\in\Delta}\{f_i^+(0)\}\ge\inf_{i\in\Delta}\{f_i^+(x)\}=(\bigwedge_{i\in\Delta}f_i^+)(x),$$

and

$$(\bigwedge_{i\in\Delta}f_i^-)(0)=\sup_{i\in\Delta}\{f_i^-(0)\}\le\sup_{i\in\Delta}\{f_i^-(x)\}=(\bigwedge_{i\in\Delta}f_i^-)(x).$$

For any $x,y\in H$, we have:

$$\begin{aligned}(\bigwedge_{i\in\Delta}f_i^+)(x) &= \inf_{i\in\Delta}\{f_i^+(x)\}\\ &\ge \inf_{i\in\Delta}\{\min\{f_i^+(x^y\,|\,x^y),f_i^+(y)\}\}\\ &= \min\{\inf_{i\in\Delta}f_i^+(x^y\,|\,x^y),\inf_{i\in\Delta}f_i^+(y)\}\\ &= \min\{(\bigwedge_{i\in\Delta}f_i^+)(x^y\,|\,x^y),(\bigwedge_{i\in\Delta}f_i^+)(y)\},\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in\Delta}f_i^-)(x) &= \sup_{i\in\Delta}\{f_i^-(x)\}\\ &\le \sup_{i\in\Delta}\{\max\{f_i^-(x^y\,|\,x^y),f_i^-(y)\}\}\\ &= \max\{\sup_{i\in\Delta}f_i^-(x^y\,|\,x^y),\sup_{i\in\Delta}f_i^-(y)\}\\ &= \max\{(\bigwedge_{i\in\Delta}f_i^-)(x^y\,|\,x^y),(\bigwedge_{i\in\Delta}f_i^-)(y)\}.\end{aligned}$$

As a result, we conclude that $\bigwedge_{i\in\Delta}f_i$ is a bipolar fuzzy SBG-ideal of the SBG-algebra H.

5. CONCLUSION

In this paper, we systematically investigated the structure of bipolar-valued fuzzy sets within the framework of Sheffer stroke BG-algebras (SBG-algebras). We introduced and formalized the concepts of bipolar-valued fuzzy SBG-subalgebras and SBG-ideals, providing rigorous definitions and characterizations. The presented algorithms offer practical methods for verifying whether a given bipolar-valued fuzzy set qualifies as an SBG-subalgebra or SBG-ideal, enhancing computational applicability. Through illustrative examples, we demonstrated the validity of our theoretical developments and provided explicit constructions of such fuzzy structures.

Furthermore, we established that the negative s-cuts and positive t-cuts of a bipolar-valued fuzzy set are closely linked to the corresponding crisp SBG-subalgebras and SBG-ideals, thus bridging the gap between fuzzy and classical algebraic concepts. The equivalence between the properties of bipolar-valued fuzzy SBG-(subalgebras/ideals) and their associated fuzzy sets was also rigorously proven. In addition, we showed that the intersection of any family of bipolar fuzzy SBG-ideals forms a bipolar fuzzy SBG-ideal, which reveals the robust lattice-theoretic nature of these structures.

The results presented here not only extend the existing theory of fuzzy algebraic structures but also provide a foundation for future research. Potential directions include the study of homomorphisms between bipolar fuzzy SBG-algebras, the investigation of quotient structures, and the application of these algebraic systems in areas such as artificial intelligence, decision theory, and information sciences. Further exploration of other types of fuzzy ideals and subalgebras, as well as the development of computational tools for their analysis, may yield deeper insights and broader applications.

Overall, this work contributes to the algebraic and fuzzy logic literature by unifying and extending the study of bipolar-valued fuzzy sets in the context of Sheffer stroke BG-algebras, and opens several avenues for future research and practical implementation.

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