Two Integral Operators Defined with Bessel Functions on the Class N(β)

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Abstract: Using Bessel functions of first kind we introduce new integral operators and show that these operators are in the class N(β).

Keywords: Analytic functions, integral operator of the first kind, Bessel function.

1. PRELIMINARY AND DEFINITIONS

Let $U = \{z : z < 1\}$ the open unit disk and A the class of all functions of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \tag{1}$$

that are analytic in *U* and satisfy the condition

$f(0) = f'(0) - 1 = 0.$

Let $N(\beta)$ be a subclass of *A* which consists all the functions $f(z)$ that satisfy the inequality:

$$Re\{\frac{zf''(z)}{f'(z)} + 1\} < \beta,\ z \in U.$$

and $M(\beta)$ be a subclass of *A* consisting of functions that satisfy the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, z \in U, \beta > 1.$$

This classes were studied by many authors, like Owa and Srivastava in [3].

The Bessel function of the first kind of order v is defined by

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+v}}{n!\Gamma(n+v+1)}.$$

The normalized Bessel function of the first kind, $f_v : U \to \mathbb{C}$ Cis defined by

$$f_v(z) = 2^v \Gamma(v+1) z^{1-v/2} J_v(z^{1/2}) = z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n!(v+1)...(v+n)}. \tag{2}$$

The Bessel functions of the first kind were studied by Szász and Kupán in [4], by Arif and Raza in [1] and also by Baricz and Frasin in [2].

To prove our main results we will use the following lemma:

Lemma 1.1 [4] Let $v > \frac{(-5+\sqrt{5})}{4}$ and consider the normalized Bessel function of the first kind $f_v : \mathbb{D} \to \mathbb{C}$, defined by $f_v(z) = 2^v \Gamma(v+1) z^{1-v/2} J_v(z^{1/2})$, where J_v stands for the Bessel function of the first kind. Then the following inequality hold for all $z \in \mathbb{D}$

$$\frac{zf_{v'}(z)}{f_v(z)} - 1 \le \frac{v+2}{4v^2 + 10v + 5}. \tag{3}$$

In this paper we introduce two integral operators using the Bessel functions of the first kind. We define:

$$I_1(f_v, g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_{v_i}(t)}{g_i(t)}\right)^{\gamma_i} dt, \tag{4}$$

and

$$I(f_v, g)(z) = \int_0^z \prod_{i=1}^n \frac{\left(f_{v_i}(t)\right)^{\gamma_i}}{\left(g_i(t)\right)^{\sigma_i}} dt, \tag{5}$$

where $f_{v_i}(z)$ are Bessel functions of the first kind and $g_i(z)$ are analytical functions.

This operator is derived from the operator defined in [5].

2. MAIN RESULTS

Theorem 2.1. Let $v_i \ge (-5+\sqrt{5}/4)$ for $i = \overline{1,n}$. If $f_{v_i}(z)$ are Bessel functions of the first kind and $g_i(z) \in M(\beta_i)$ for $\beta_i > 1$, then the operator $I(f_v, g)(z))$ is

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in the class $N(\eta)$, where $\eta = 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + `0\nu_i + 5}\right) + \sum_{i=1}^{n}\sigma_i\beta_i > 1$ for $i = \overline{1,n}$.

Proof. From the definition of the class $N(\beta)$ it follows that

$$Re\left(1 + \frac{zI''(f_\nu, g)(z))}{I'(f_\nu, g)(z))}\right) = 1 + \sum_{i=1}^{n}\gamma_i Re\left(\frac{zf_{\nu_i}'(z)}{f_{\nu_i}(z)}\right)$$
$$- \sum_{i=1}^{n}\sigma_i Re\left(\frac{zg_{i'}(z)}{g_i(z)}\right)$$

Because $f_{\nu_i}(z)$ are Bessel functions of the first kind it follows from Lemma 1.1 that

$$Re\left(\frac{zf_{\nu_i}'(z)}{f_{\nu_i}(z)}\right) \le 1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}.$$

Using this we obtain that:

$$Re\left(1 + \frac{zI''(f_\nu, g)(z))}{I'(f_\nu, g)(z))}\right) \le 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}\right) + \sum_{i=1}^{n}\sigma_i\beta_i.$$

So, it follows that $I(f_\nu, g)(z)) \in N(\eta)$, where $\eta = 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}\right) + \sum_{i=1}^{n}\sigma_i\beta_i$, for $i = \overline{1,n}$.

Corollary 2.1. If $f_\nu(z)$ are Bessel functions of the first kind and $g_i(z) \in M(\beta_i)$ for $\beta_i > 1$, then the operator $I(f_\nu, g)(z)) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_\nu(t)}{g_i(t)}\right)^{\gamma_i}$ is in the class $N(\eta)$, where $\eta = 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\right) + \sum_{i=1}^{n}\sigma_i\beta_i > 1$ for $i = \overline{1,n}$.

Proof. We consider $\nu_1 = \nu_2 = \ldots = \nu_n = \nu$ in Theorem 2.2.

Theorem 2.2. Let $\nu_i > (-5 + \sqrt{5})/4$ for $i = 1, \ldots, n$ .If $f_{\nu_i}(z)$ are Bessel functions of the first kind defined by $f_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i + 1)z^{1-\nu_i/2}J_{\nu_i}(z^{1/2})$ and $g_i(z) \in M(\alpha_i)$ for $\alpha_i > 1$, then the operator $I_1(f_\nu, g)(z)$ is in the class $N(\theta)$, where

$$\theta = 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} + \alpha_i\right) > 1$$

Proof. Using the fact that the operator $I_1(f_\nu, g)(z)$ is in the class $N(\theta)$ it follows that:

$$Re\left(1 + \frac{zI_{1''}(f_\nu, g)(z)}{I_{1'}(f_\nu, g)(z)}\right) = 1 + \sum_{i=1}^{n}\gamma_i Re\left(\frac{zf_{\nu_i}'(z)}{f_{\nu_i}(z)}\right)$$
$$- \sum_{i=1}^{n}\gamma_i Re\left(\frac{zg_{i'}(z)}{g_i(z)}\right) \tag{6}$$

Using Lemma 1.1 we obtain that

$$Re\left(\frac{zf_{\nu_i}'(z)}{f_{\nu_i}(z)}\right) \le 1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}.$$

Because $g_i(z) \in M(\alpha_i)$ it follows that $Re\left(\frac{zg_{i'}(z)}{g_i(z)}\right) < \alpha_i$. Using the above relations it follows that the relation (6) is equivalent with

$$Re\left(1 + \frac{zI_{1''}(f_\nu, g)(z)}{I_{1'}(f_\nu, g)(z)}\right) \le 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} + \alpha_i\right),$$

so the operator $I_1(f_\nu, g)(z) \in N(\theta)$ where $\theta = 1 + \sum_{i=1}^{n}\gamma_i\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} + \alpha_i\right) > 1$ for $i = 1, \ldots, n$.

Corollary 2.2. Let $\nu_i > (-5 + \sqrt{5})/4$ for $i = 1, \ldots, n$ .If $f_{\nu_i}(z)$ are Bessel functions of the first kind defined by $f_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i + 1)z^{1-\nu_i/2}J_{\nu_i}(z^{1/2})$ and $g_i(z)$ are starlike functions of order α_i, then the operator

$$I(f_\nu, g)(z) = \int_0^z \prod_{i=1}^{n}\frac{f_{\nu_i}(t)}{g_i(t)}dt$$

is in the class $N(\theta)$, where

$$\theta = 1 + \sum_{i=1}^{n}\left(1 + \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} + \alpha_i\right) > 1$$

Proof. We consider $\gamma_1 = \ldots = \gamma_n = 1$ in Theorem 2.2.

Corollary 2.3. Let $\nu > (-5 + \sqrt{5})/4$. If $f_\nu(z)$ are Bessel functions of the first kind defined by $f_\nu(z) = 2^{\nu}\Gamma(\nu + 1)z^{1-\nu/2}J_\nu(z^{1/2})$ and $g_i(z) \in M(\alpha)$ for $\alpha > 1$, then the operator $I_1(f_\nu, g)(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_\nu(t)}{g_i(t)}\right)^{\gamma}$ is in the class $N(\theta)$, where

$$\theta = 1 + n\gamma\left(1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5} + \alpha\right) > 1.$$

Proof. We consider $v_1 = ... = v_n = v, \alpha_1 = \alpha_2 ... = \alpha_n = \alpha$ and $\gamma_1 = ... = \gamma_n = \gamma$ in Theorem 2.2.

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